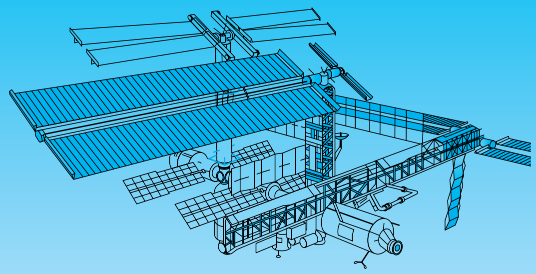
CHAPTER 9



Fourier Series

When analyzing situations as diverse as electrical oscillations, vibrating mechanical systems, longitudinal oscillations in crystals, and many other physical phenomena, Fourier series are found to arise naturally. Furthermore, the individual terms in a Fourier series often have an important physical interpretation. In a vibrating mechanical system, for example, each component of a Fourier series representation of the overall vibration represents a fundamental mode of vibration. The full Fourier series shows how each mode contributes to the solution, and which are the most significant modes. This information can often be used to advantage, either by showing how the modes can be utilized to achieve a desired effect, or by using the information to enable systems to be constructed that minimize undesirable vibrations. It is for these and other reasons that it is necessary for engineers and physicists to study the properties of Fourier series.

9.1 Introduction to Fourier Series

A **Fourier series** representation of a function $f(x)$ over the interval $-\pi \leq x \leq \pi$ is an expression of the form

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \\ &= a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots, \end{aligned} \tag{1}$$

where the coefficients $a_0, a_1, \ldots, b_1, b_2, \ldots$ are determined by the function $f(x)$.

It is important to notice that the Fourier series representation of $f(x)$ contains two infinite sums, one of even functions (the cosines) and the other of odd functions (the sines). It will be recalled that a function $f(x)$ defined in the interval $-L \leq x \leq L$ is said to be an **even function** in the interval if

even and odd function

$$f(-x) = f(x), \tag{2}$$

and to be an **odd function** in the interval if

$$f(-x) = -f(x). \tag{3}$$

The cosine function is an even function because $\cos(-x) = \cos x$ in agreement with the definition in (2). As this is true for all x, the function $\cos x$ is an even function for $-\infty < x < \infty$. Similarly, $\sin x$ is an odd function because $\sin(-x) = -\sin x$ in agreement with the definition in (3). This also is true for all x, so the function $\sin x$ is an odd function for $-\infty < x < \infty$.

Most functions are neither even nor odd, but any function in an interval $-L \leq x \leq L$ can be expressed as the sum of an even function and an odd function defined over the interval. To see why this is, let $f(x)$ be an arbitrary function defined over the interval $-L \leq x \leq L$, and write it in the form

$$f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) \quad \text{for } -L \leq x \leq L. \tag{4}$$

Then the function

$$h(x) = \frac{1}{2}(f(x) + f(-x)) \tag{5}$$

is seen to be an *even* function, because $h(-x) = h(x)$, whereas the function

$$g(x) = \frac{1}{2}(f(x) - f(-x)) \tag{6}$$

is seen to be an odd function, because $g(-x) = -g(x)$, so the assertion is proved.

EXAMPLE 9.1

Classify the following functions as even, odd, or neither.

(a) $\cosh x$. (b) $\sinh x$. (c) $x^2 + \sin x$. (d) $1 + x^2 + 3x^4$.

Solution (a) As $\cosh(-x) = \cosh x$ for all x, the function $\cosh x$ is an even function for all x. (b) As $\sinh(-x) = -\sinh x$ for all x, the function $\sinh x$ is an odd function for all x. (c) $(-x)^2 = x^2$, so x^2 is an even function for all x, while $\sin x$ is an odd function for all x, so the function $x^2 + \sin x$ is neither even nor odd. In this case the function $x^2 + \sin x$ is already expressed as the sum of an even function and an odd function. (d) Set $f(x) = 1 + x^2 + 3x^4$. Then $f(-x) = 1 + (-x)^2 + (-x)^4 = f(x)$, so $f(x)$ is an even function. This result can be obtained by a different form of argument as follows. A constant does not change when the sign of x is changed, so all constants are even functions and, in particular, 1 is an even function. The function x^2 has already been shown to be an even function, and the function $3x^4$ is an even function because $3(-x)^4 = 3x^4$. Thus, as the function $1 + x^2 + 3x^4$ is a sum of three even functions, it must be an even function. ■

deriving formulas for a_n and b_n

To arrive at a formula for the a_n in (1) corresponding to a given function $f(x)$, result (1) is first multiplied term by term by $\cos nx$ to obtain

$$\begin{aligned} f(x)\cos nx = {} & a_0 \cos nx + a_1 \cos x \cos nx + a_2 \cos 2x \cos nx + a_3 \cos 3x \cos nx \\ & + \cdots + a_{n-1} \cos(n-1)x \cos nx + a_n \cos^2 nx \\ & + a_{n+1} \cos(n+1)x \cos nx + \cdots + b_1 \sin x \cos nx \\ & + b_2 \sin 2x \cos nx + \cdots . \end{aligned}$$

Integrating this result over the interval $-\pi \le x \le \pi$ gives

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos nx dx &= a_0\int_{-\pi}^{\pi}\cos nx dx + a_1\int_{-\pi}^{\pi}\cos x\cos nx dx\\ &\quad + a_2\int_{-\pi}^{\pi}\cos 2x\cos nx dx + a_3\int_{-\pi}^{\pi}\cos 3x\cos nx dx + \cdots\\ &\quad + a_{n-1}\int_{-\pi}^{\pi}\cos(n-1)x\cos nx dx + a_n\int_{-\pi}^{\pi}\cos^2 nx dx\\ &\quad + a_{n+1}\int_{-\pi}^{\pi}\cos(n+1)x\cos nx dx + \cdots + b_1\int_{-\pi}^{\pi}\sin x\cos nx dx\\ &\quad + b_2\int_{-\pi}^{\pi}\sin 2x\cos nx dx + \cdots.\end{aligned}$$

The orthogonality properties of the sine and cosine functions listed in entry 1 of the summary of main sets of orthogonal functions in Section 8.11 shows that all integrals on the right with the exception of the one with the integrand $\cos^2 nx$ vanish, giving rise to the result

$$\int_{-\pi}^{\pi} f(x)\cos nx dx = a_n\int_{-\pi}^{\pi}\cos^2 nx dx.$$

However, $\int_{-\pi}^{\pi}\cos^2 nx dx = \pi$, for $n \ne 0$ and $\int_{-\pi}^{\pi} 1.dx = 2\pi$, so

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \quad \text{and} \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad \text{for } n = 1, 2, \ldots.$$

A similar argument involving the multiplication of the Fourier series (1) by $\sin nx$ followed by integration over the interval $-\pi \le x \le \pi$ and use of the orthogonality properties of $\sin nx$ shows the coefficients b_n are given by

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad \text{for } n = 1, 2, \ldots.$$

the Euler formulas

These results are the **Euler formulas** for the **Fourier coefficients** a_n and b_n, and for future reference they are now listed, together with the associated Fourier series representation of $f(x)$.

the Fourier series representation

Fourier series representation of $f(x)$ over the interval $-\pi \le x \le \pi$

Let the function $f(x)$ be defined on the interval $-\pi \le x \le \pi$. Then the Fourier coefficients a_n and b_n in the Fourier series representation of $f(x)$

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \tag{7}$$

are given by the Euler formulas

$$\begin{aligned}&a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx,\\ &b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \ldots \quad \text{for } n = 1, 2, \ldots.\end{aligned} \tag{8}$$

The arguments used to derive the Euler formulas in (8) are not rigorous, because the term by term integration needs to be justified and the convergence of the Fourier series representation of $f(x)$ to the function $f(x)$ itself has not been examined, so the use of an equality sign in (1) and (7) must be questioned.

JEAN BAPTISTE JOSEPH (BARON) FOURIER (1768–1830)
A remarkable French physicist who was also an outstanding mathematician. He was orphaned at eight, and educated in a military school run by the Benedictines who then gave him a lectureship in mathematics. He later moved to a chair at the Ecole Polytechnique in Paris, and later to Grenoble where he was appointed Prefect by Napoleon. His experiments on heat conduction while in Grenoble, suggested by Newton's Law of Cooling, led him to propose his law of heat conduction (Fourier's Law) and to the publication of his most important *Theorie Analytique de la Chaleur* in which he introduced the representation of arbitrary function over an interval in terms of trigonometric functions, now called Fourier series. He was created a Baron by Napoleon in 1808.

In fact, the preceding approach can be fully justified for all functions $f(x)$ that arise in practical situations, and we will see later that the equality sign can be used wherever $f(x)$ is continuous, whereas at points where $f(x)$ experiences a finite jump discontinuity the value assumed by the Fourier series representation is the average of the values to the immediate left and right of the jump. It is for these reasons that in more advanced accounts the equality sign in (7) is replaced by a tilde $\sim$, because this indicates that a relationship exists between a function $f(x)$ and its Fourier series representation without indicating that it is necessarily a strict equality. When this notation is used, the connection between $f(x)$ and its Fourier series is shown by writing

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{9}$$

fundamental interval, periodicity, and periodic extension

The interval of integration $-\pi \leq x \leq \pi$ used when deriving the Euler formulas is called the **fundamental interval** of the Fourier series, and the Fourier coefficients will always be defined provided the integral $\int_{-\pi}^{\pi} f(x)dx$ exists. Although Fourier series comprise only even and odd functions, results (4) to (6) allow a Fourier series to represent arbitrary functions that are neither even nor odd.

A Taylor series expansion of a function $f(x)$ about a point x_0 requires the function to be repeatedly differentiable at x_0. However, the coefficients of a Fourier series are defined in terms of definite integrals that are still defined when $f(x)$ has finite jump discontinuities in the fundamental interval, so the Euler formulas still remain valid when $f(x)$ is discontinuous. It is this property of a definite integral that makes a Fourier series representation of a function more general than a Taylor series expansion.

The properties of Fourier series reflect the *periodicity* of the sine and cosine functions used in the expansion, where the *period* of a periodic function is defined as follows. A function $g(x)$ is said to be **periodic** with **period** T if

$$g(x+T) = g(x) \tag{10}$$

for all x, and T is the *smallest* value for which (10) is true. A periodic function $g(x)$ may either be continuous or discontinuous, and an example of a continuous periodic function with period T is shown in Fig. 9.1.

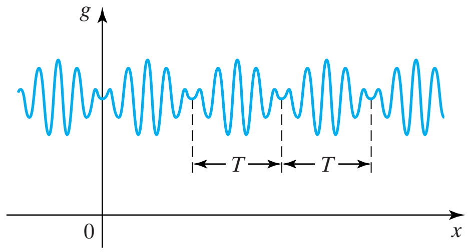


FIGURE 9.1 A continuous periodic function $g(x)$ with period T.

The functions 1, $\cos nx$, and $\sin nx$ in the Fourier series representation (7) of $f(x)$ are all periodic with period 2π, so the *Fourier series representation* of $f(x)$ defined on the interval $-\pi < x < \pi$ is also periodic with period 2π. It does not necessarily follow that outside the fundamental interval the function $f(x)$ coincides with its Fourier series representation, because the behavior of $f(x)$ outside the fundamental interval does not enter into the Euler formulas. Each representation of $f(x)$ in an interval of the form $(2n-1)\pi < x < (2n+1)\pi$, with $n = 0, \pm 1, \pm 2, \ldots$, is called a **periodic extension** of the fundamental interval for $f(x)$.

In Chapter 8, Example 8.22, the discontinuous rectangular pulse function

$$f(x) = \begin{cases} 0, & -\pi < x < -\pi/2 \\ 1, & -\pi/2 < x < \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$$

was shown to be represented by the Fourier series

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \frac{\cos 7x}{7} + \cdots\right] \quad \text{for all } x. \tag{11}$$

If this function $f(x)$ is defined for all x by the periodicity condition $f(x + 2\pi) = f(x)$, its graph takes the form shown in Fig. 9.2. Figure 9.3 shows the graph of the first five terms of the Fourier series representation (11) in the fundamental interval.

This simple example emphasizes two important issues that always arise when working with Fourier series representations of functions:

1. The need to interpret the equality sign in (7) at any point $x = x_0$ in the fundamental interval where $f(x)$ is discontinuous.
2. The fact that the Fourier series of a function and the periodic extensions of the function will only coincide when the function $f(x)$ is itself periodic with a period equal to the fundamental interval.

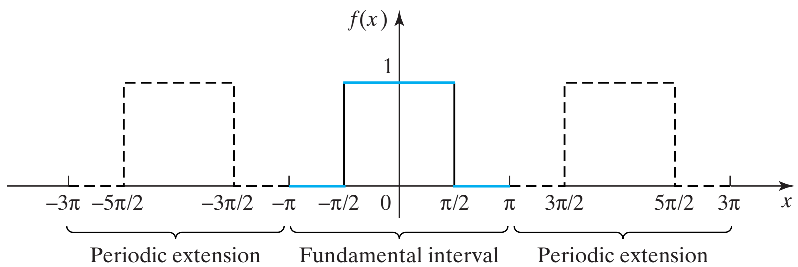


FIGURE 9.2 The periodic rectangular pulse function $f(x)$.

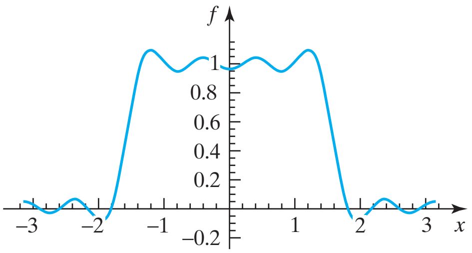


FIGURE 9.3 Graph of the first five terms of the Fourier series of $f(x)$.

An example of the difference that can arise between the behavior of a nonperiodic function $f(x)$ and its periodic extensions is illustrated in Fig. 9.4 in the case of the function

$$f(x) = \begin{cases} 1/2, & x < -\pi \\ 0, & -\pi < x < -\pi/2 \\ 1, & -\pi/2 < x < \pi/2 \\ 0, & \pi/2 < x < \pi \\ 1/4, & x > \pi. \end{cases}$$

The periodic extensions of $f(x)$ in its fundamental interval $-\pi \le x \le \pi$ shown as dashed lines are, of course, the same as those in Fig. 9.2, though in this case the behavior of $f(x)$ outside the fundamental interval is entirely different.

EXAMPLE 9.2

some illustrative examples

Find the Fourier series representation of

$$f(x) = \begin{cases} \sin 2x, & -\pi < x < -\pi/2 \\ 0, & -\pi/2 \le x \le 0 \\ \sin 2x, & 0 < x \le \pi. \end{cases}$$

Solution The function $f(x)$ is continuous over the fundamental interval $-\pi \le x \le \pi$, but it is defined in piecewise manner, so the Fourier coefficients must be determined by integrating the Euler equations (8) in a corresponding manner. We have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{-\pi/2} \sin 2x dx + \frac{1}{2\pi}\int_0^{\pi} \sin 2x dx$$
$$= \frac{1}{2\pi}[-(1/2)\cos 2x]_{-\pi}^{-\pi/2} + \frac{1}{2\pi}[-(1/2)\cos 2x]_0^{\pi} = \frac{1}{2\pi} + 0 = \frac{1}{2\pi}.$$

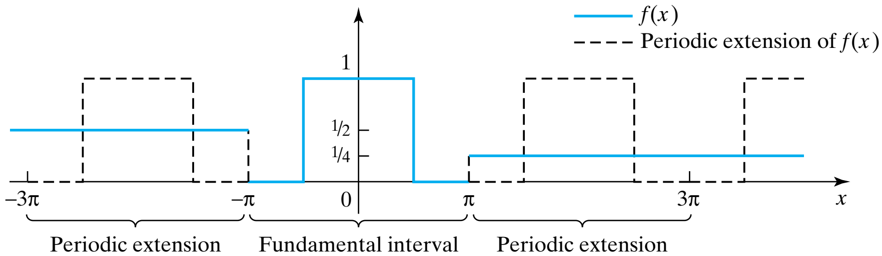


FIGURE 9.4 A nonperiodic function defined for all x, and the periodic extensions of the function in its fundamental interval.

Similarly,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{-\pi/2} \sin 2x \cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} \sin 2x \cos nx dx$$

$$= \frac{-2}{\pi}\left[\frac{\cos n\pi + \cos(n\pi/2)}{n^2-4}\right]_{-\pi}^{-\pi/2} + \frac{2}{\pi}\left[\frac{\cos n\pi - 1}{n^2-4}\right]_{0}^{\pi}, \quad \text{for } n \neq 2$$

$$= \frac{-2[1+\cos(n\pi/2)]}{\pi(n^2-4)}, \quad \text{for } n \neq 2.$$

As the denominator in the expression for a_n is zero when $n = 2$, in order to find a_2 it is necessary to return to the Euler formula for a_n and set $n = 2$ *before* integrating, when we obtain

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{-\pi/2} \sin 2x \cos 2x dx + \frac{1}{\pi}\int_{0}^{\pi} \sin 2x \cos 2x dx = 0 + 0 = 0.$$

The Euler formula for b_n becomes

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{-\pi/2} \sin 2x \sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} \sin 2x \sin nx dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-2)x}{n-2} - \frac{\sin(n+2)x}{n+2}\right]_{-\pi}^{-\pi/2} + \frac{1}{2\pi}\left[\frac{\sin(n-2)x}{n-2} - \frac{\sin(n+2)x}{n+2}\right]_{0}^{\pi}$$

$$= \frac{2}{\pi}\frac{\sin(n\pi/2)}{(n^2-4)}, \quad \text{for } n \neq 2.$$

As the denominator in the expression for b_n is zero for $n = 2$, to find b_2 we must set $n = 2$ in the Euler formula for b_2 before integrating, as a result of which we find that

$$b_2 = \frac{1}{\pi}\int_{-\pi}^{-\pi/2} \sin^2 2x dx + \frac{1}{\pi}\int_{0}^{\pi} \sin^2 2x dx$$

$$= \frac{1}{4\pi}[2x - \sin 2x \cos 2x]_{-\pi}^{-\pi/2} + \frac{1}{4\pi}[2x - \sin 2x \cos 2x]_{0}^{\pi}$$

$$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

Combining the preceding results shows the first few Fourier coefficients to be

$$a_0 = \frac{1}{2\pi}, \quad a_1 = \frac{2}{3\pi}, \quad a_2 = 0, \quad a_3 = -\frac{2}{5\pi}, \quad a_4 = -\frac{1}{3\pi}, \quad a_5 = -\frac{2}{21\pi},$$

$$b_1 = -\frac{2}{3\pi}, \quad b_2 = \frac{3}{4}, \quad b_3 = -\frac{2}{5\pi}, \quad b_4 = 0, \quad b_5 = \frac{2}{21\pi}, \cdots.$$

When these coefficients are used, the first few terms of the Fourier series for $f(x)$ are seen to be

$$f(x) = \frac{1}{2\pi} + \frac{1}{\pi}\left(\frac{2}{3}\cos x - \frac{2}{5}\cos 3x - \frac{1}{3}\cos 4x - \frac{2}{21}\cos 5x + \cdots\right)$$

$$+ \frac{1}{\pi}\left(-\frac{2}{3}\sin x + \frac{3\pi}{4}\sin 2x - \frac{2}{5}\sin 3x + \frac{2}{21}\sin 5x + \cdots\right).$$

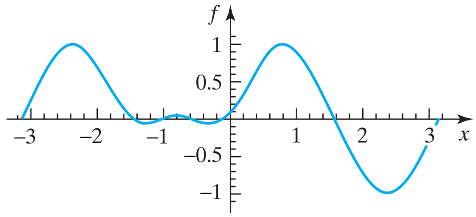


FIGURE 9.5 Fourier series approximation for $f(x)$.

This example illustrates how when a sine function (or a cosine function) with an argument mx with m an integer occurs in a piecewise defined function, its Fourier coefficients a_m and b_m must be found from the Euler formulas with n set equal to m before integration. Figure 9.5 shows a graph of this Fourier series approximation to $f(x)$ up to and including the terms in $\cos 5x$ and $\sin 5x$. ■

It is useful to have a special name for finite approximations to Fourier series such as the one used to construct the graph in Fig. 9.5. Because of this it is usual to call the approximation

$$S_N(x) = a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx) \tag{12}$$

Nth partial sum

to the full Fourier series in (7) the ***N*th partial sum** of the Fourier series. Thus, the graph in Fig. 9.5 shows the fifth partial sum $S_5(x)$ of the function $f(x)$ defined in Example 9.2. The Fourier series in (7) is related to its Nth partial sum $S_n(x)$ by the limit

$$f(x) = a_0 + \sum_{n=0}^{\infty}(a_n \cos nx + b_n \sin nx) = \lim_{N\to\infty} S_N(x). \tag{13}$$

Not every function has a Fourier series involving an infinite number of terms, as can be seen by considering the function $f(x) = 1 + 2\sin x \cos x$. When this is rewritten as $f(x) = 1 + \sin 2x$, it is recognized that it is, in fact, its own Fourier series.

There is nothing special about the choice of $-\pi \le x \le \pi$ as a fundamental interval, and it is often necessary to take the fundamental interval to be $-L \le x \le L$. Results (7) and (8) generalize immediately once it is recognized that the set of functions

$$1, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \cos\frac{3\pi x}{L}, \ldots, \sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \sin\frac{3\pi x}{L}, \ldots$$

form an orthogonal set over the interval $-L \le x \le L$. This can be seen by using routine integration to show that

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0 \quad \text{for all integers } m \text{ and } n, \tag{14}$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0 & \text{for } m \ne n \\ L & \text{for } m = n \end{cases} \quad \text{for all integers } m \text{ and } n, \tag{15}$$

and

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = \begin{cases} 0 & \text{for } m \neq n \\ L & \text{for } m = n \neq 0 \;\; \text{for all integers } m \text{ and } n \\ 2L & \text{for } m = n = 0. \end{cases} \tag{16}$$

The Fourier series of a function $f(x)$ defined on the interval $-L \leq x \leq L$ becomes

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right), \tag{17}$$

and the corresponding Euler formulas for the a_n and b_n follow as before. The coefficients a_n are obtained by multiplying (17) by $\cos\frac{n\pi x}{L}$ and integrating over the interval $-L \leq x \leq L$, while the b_n follow by multiplying (17) by $\sin\frac{n\pi x}{L}$ and integrating over the same interval. The result is as follows, though the details are left as an exercise.

Fourier series over $-L \leq x \leq L$

Fourier series representation of $f(x)$ over the interval $-L \leq x \leq L$

Let the function $f(x)$ be defined on the interval $-L \leq x \leq L$. Then the Fourier coefficients a_n and b_n in the Fourier series representation of $f(x)$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{18}$$

are given by the Euler formulas

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx,$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad \text{for } n = 1, 2, \ldots. \tag{19}$$

EXAMPLE 9.3 Find the Fourier series representation of $f(x) = x + 1$ for $-1 \leq x \leq 1$.

Solution In this case $L = 1$, so using integration by parts we find that

$$a_0 = \frac{1}{2}\int_{-1}^{1}(x+1)dx = 1, \quad a_n = \int_{-1}^{1}(x+1)\cos n\pi x\,dx = \frac{\cos n\pi x}{n^2\pi^2} + \frac{x\sin n\pi x}{n\pi} + \left.\frac{\sin n\pi x}{n\pi}\right]_{-1}^{1} = 0$$

and

$$b_n = \int_{-1}^{1}(x+1)\sin n\pi x\,dx = \frac{\sin n\pi x}{n^2\pi^2} - \frac{x\cos n\pi x}{n\pi} - \left.\frac{\cos n\pi x}{n\pi}\right]_{-1}^{1} = \frac{2(-1)^{n+1}}{n\pi},$$

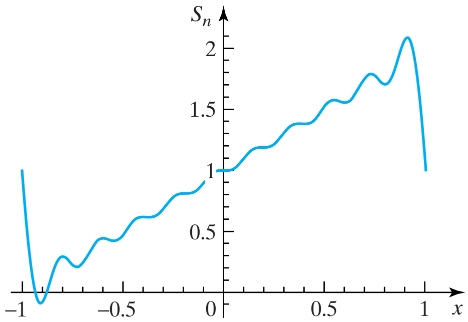


FIGURE 9.6 The partial sum approximation $S_{10}(x)$.

for $n = 1, 2, \ldots,$ where we have used the fact that $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$ for n a positive integer. Substituting these coefficients into (18) shows the required Fourier series representation to be

$$f(x) = 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi x, \quad \text{for } -1 \le x \le 1.$$

A graph of the partial sum approximation $S_{10}(x)$ to $f(x)$ is shown in Fig. 9.6. ■

As cosines are even functions and sines are odd functions, it is to be expected that a Fourier series representation of an even function will only contain cosine terms, whereas a Fourier series representation of an odd function will only contain sine functions. These properties form the basis of the following result that simplifies the task of finding Fourier series representations of even and odd functions.

expanding even and odd functions

Fourier series of even and odd functions

If $f(x)$ is an even function defined on the interval $-L \le x \le L$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad \text{with } a_0 = \frac{1}{L}\int_0^L f(x)dx,$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

for $n = 1, 2, \ldots;$ if $f(x)$ is an odd function, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \quad \text{with } b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx,$$

$$\text{for } n = 1, 2, \ldots,$$

The justification of these results is as follows. To find the form taken by the Fourier coefficients a_n of an even function, and why its Fourier coefficients b_n vanish, we will consider an even function $f(x)$ defined over the interval $-L \le x \le L$.

By definition,

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{2L}\int_{-L}^{0} f(x)dx + \frac{1}{2L}\int_{0}^{L} f(x)dx.$$

Setting $x = -u$ in the first integral on the right gives

$$\frac{1}{2L}\int_{-L}^{0} f(x)dx = -\frac{1}{2L}\int_{L}^{0} f(-u)du.$$

As f is an even function, $f(-u) = f(u)$, so using this result, changing the sign of the integral by interchanging its limits, and then replacing the dummy variable u by x gives

$$\frac{1}{2L}\int_{-L}^{0} f(x)dx = \frac{1}{2L}\int_{0}^{L} f(x)dx.$$

When this is combined with the original expression for a_0 we find that

$$a_0 = \frac{1}{L}\int_{0}^{L} f(x)dx,$$

and a strictly analogous argument shows that

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos n\pi x dx \quad \text{for } n = 1, 2, \ldots.$$

The Fourier coefficients b_n are given by

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}dx + \frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx.$$

Setting $x = -u$ in the integral taken over the interval $-L \le x \le 0$ gives

$$\frac{1}{L}\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}dx = -\frac{1}{L}\int_{L}^{0} f(-u)\sin\left(-\frac{n\pi u}{L}\right)du.$$

We now use the fact that f is an even function, so $f(-u) = f(u)$, together with the fact that the sine function is an odd function. Reversal of the limits coupled with changing the sign and replacing u by x gives

$$\frac{1}{L}\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}dx = -\frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx.$$

Finally, using this result in the original expression for b_n gives

$$b_n = \frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx - \frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx = 0 \quad \text{for } n = 1, 2, \ldots,$$

and the result is proved.

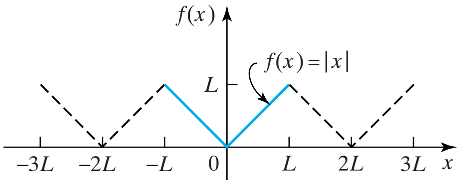


FIGURE 9.7 The function $f(x) = |x|$ in $-L \leq x \leq L$ and two periodic extensions.

A similar argument shows that if $f(x)$ is an odd function over $-L \leq x \leq L$, then

$$a_n = 0 \quad \text{for } n = 0, 1, 2, \ldots,$$

and

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} dx \quad \text{for } n = 1, 2, \ldots,$$

and the results have been established.

EXAMPLE 9.4 Find the Fourier series representation of $f(x) = |x|$ in the interval $-L \leq x \leq L$.

Solution The graph of this even function, together with two of its periodic extensions outside the fundamental interval $-L \leq x \leq L$, is shown in Fig. 9.7.

The Euler formula for the coefficients a_n of the *even* function $|x|$ defined as

$$|x| = \begin{cases} -x & \text{for } < 0 \\ x & \text{for } x \geq 0 \end{cases}$$

gives

$$a_0 = \frac{1}{L}\int_0^L x dx = \frac{L}{2}$$

and

$$a_n = \frac{2}{L}\int_0^L x \cos\frac{n\pi x}{L} dx = \frac{2}{L}\left[\frac{L^2 \cos\frac{n\pi x}{L}}{n^2\pi^2} + \frac{Ln\pi x \sin\frac{n\pi x}{L}}{n^2\pi^2}\right]_0^L, \quad \text{for } n = 1, 2, \ldots.$$

a convenient representation of cos $n\pi$

If we use the fact that $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$ when n is a positive integer, it then follows that

$$a_n = \frac{2L}{n^2\pi^2}[(-1)^n - 1] \quad \text{for } n = 1, 2, \ldots,$$

and so

$$a_n = -\frac{4L}{n^2\pi^2} \quad \text{when } n \text{ is odd}$$

and

$$a_n = 0 \quad \text{when } n \neq 0, \text{ is even.}$$

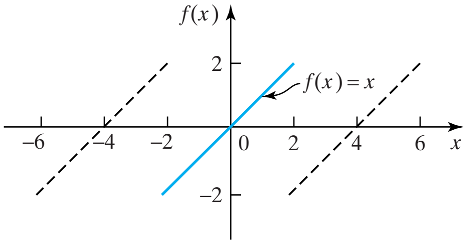


FIGURE 9.8 The function $f(x) = x$ in $-2 \le x \le 2$ and two periodic extensions.

Thus, the Fourier series representation of $f(x) = |x|$ for $-L \le x \le L$ is

$$f(x) = \frac{L}{2} - \frac{4L}{\pi^2}\left(\frac{\cos\dfrac{\pi x}{L}}{1^2} + \frac{\cos\dfrac{3\pi x}{L}}{3^2} + \frac{\cos\dfrac{5\pi x}{L}}{5^2} + \cdots\right).$$

The sequence of positive odd numbers can be written in the form $2n - 1$ with $n = 1, 2, \ldots,$ so this last result can be expressed more concisely as

$$f(x) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\left(\dfrac{(2n-1)\pi x}{L}\right)}{(2n-1)^2} \quad \text{for } -L \le x \le L. \qquad \blacksquare$$

EXAMPLE 9.5 Find the Fourier series representation of $f(x) = x$ on the interval $-2 \le x \le 2$.

Solution A graph of $f(x)$ and two of its periodic extensions outside the fundamental interval $-2 \le x \le 2$ is shown in Fig. 9.8.

Using the fact that $L = 2$, a straightforward calculation gives

$$b_n = \frac{1}{2}\int_{-2}^{2} x \sin\frac{n\pi x}{2}\,dx = \frac{2}{n^2\pi^2}\left[\sin\frac{n\pi x}{2} - \frac{1}{2}n\pi x\cos\frac{n\pi x}{2}\right]_{-2}^{2}$$

$$= -\frac{4\cos n\pi}{n\pi} = \frac{4(-1)^{n+1}}{n\pi},$$

and as the function is odd all the coefficients $a_n = 0$.

The required Fourier series representation is thus

$$f(x) = \frac{4}{\pi}\left(\frac{\sin\dfrac{\pi x}{2}}{1} - \frac{\sin \pi x}{2} + \frac{\sin\dfrac{3\pi x}{2}}{3} - \cdots\right),$$

which can be written in the more concise form

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{2} \quad \text{for } -2 \le x \le 2. \qquad \blacksquare$$

Summary

Fourier series have been defined over more general intervals than $-\pi \le x \le \pi$ and the notion of a periodic extension has been introduced. Attention has been drawn to the behavior of a Fourier series representation at a point of discontinuity of $f(x)$, and the expansion of even and odd functions has been considered.

EXERCISES 9.1

Find the period of each of the functions in Exercises 1 through 6.

1. $\cos x + \sin 2x$.

2. $2\sin 2x - 3\cos \dfrac{x}{3}$.

3. $\sin x \cos x$.

4. $\cos 2x \sin x$.

5. $3\sin \dfrac{x}{3} + \cos \dfrac{x}{2}$.

6. $\cos \dfrac{x}{3} + 5\sin \dfrac{x}{4}$.

In Exercises 7 through 10 (a) sketch the given function in the interval $-3a < x < 3a$, and (b) in the intervals $-3a < x < -a$ and $a < x < 3a$, and state whether the function is periodic.

7. $f(x) = \begin{cases} 0, & x < a/2 \\ 1, & x > a/2. \end{cases}$

8. $f(x) = \begin{cases} -1, & -a < x < 0 \\ 2, & 0 < x < a, \end{cases} \quad f(x+2a) = f(x).$

9. $f(x) = a - |x|$.

10. $f(x) = |\sin \pi x/a|$.

In Exercises 11 and 12 make use of the trigonometric identities $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ and $\cos(A \pm B) = \cos A \cos B + \sin A \sin B$ to transform the given functions into their (finite) Fourier series.

11. (a) $\sin x \cos x$. (b) $1 - 2\sin^2 x$. (c) $\sin 3x \cos x$.

12. (a) $4\cos 2x \cos 5x$. (b) $\sin x \sin 2x$. (c) $\cos^2 2x - 1/2$.

Verify the following definite integrals that were used when developing a Fourier series representation over the interval $-L < x < L$.

13. $\displaystyle\int_{-L}^{L} \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx = 0$ for all integers m and n.

14. $\displaystyle\int_{-L}^{L} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx = \begin{cases} 0 & \text{for } m \ne n \\ L & \text{for } m = n, \\ & \text{with } m, n \text{ integers.} \end{cases}$

15. $\displaystyle\int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx = \begin{cases} 0 & \text{for } m \ne n \\ L & \text{for } m = n \ne 0 \\ 2L & \text{for } m = n = 0 \text{ for all integers } m \text{ and } n. \end{cases}$

16. Prove that the product of two even functions and of two odd functions is an even function, and that the product of an even and an odd function is an odd function.

17. Prove that the sum of two even functions is an even function and the sum of two odd functions is an odd function.

18. Prove that if $f(x)$ is an odd function all the Fourier coefficients $a_n = 0$.

19. Evaluate the following integrals that arise when finding the Fourier series expansion of x over the interval $-L < x < L$.

(a) $\displaystyle\int_{-L}^{L} x \sin \frac{\pi x}{L} dx.$ **(b)** $\displaystyle\int_{-L}^{L} x \sin \frac{2\pi x}{L} dx.$

(c) $\displaystyle\int_{-L}^{L} x \sin \frac{3\pi x}{L} dx.$

20. Evaluate the following integrals that arise when finding the Fourier series expansion of x^2 over the interval $-L < x < L$.

(a) $\displaystyle\int_{-L}^{L} x^2 \sin \frac{\pi x}{L} dx.$ **(b)** $\displaystyle\int_{-L}^{L} x^2 \sin \frac{2\pi x}{L} dx.$

(c) $\displaystyle\int_{-L}^{L} x^2 \sin \frac{3\pi x}{L} dx.$

The integrals in Exercises 21 and 22 arise when finding the Fourier series expansion of e^{ax} over the interval $-L < x < L$. Use the result $\cos n\pi = (-1)^n$ for integral values of n to establish the stated result.

21. $\displaystyle\int_{-\pi}^{\pi} e^{ax} \sin nx dx = (-1)^{n+1} \frac{n(e^{a\pi} - e^{-a\pi})}{(a^2 + n^2)}$ for integral values of n.

22. $\displaystyle\int_{-\pi}^{\pi} e^{ax} \cos nx dx = (-1)^{n} \frac{a(e^{a\pi} - e^{-a\pi})}{(a^2 + n^2)}$ for integral values of n.

In Exercises 23 through 35 find the Fourier series representation of the given function over the indicated fundamental interval and use a computer to plot the indicated partial sum $S_n(x)$ over the fundamental interval.

23. $f(x) = \begin{cases} a, & -\pi < x < 0 \\ b, & 0 < x < \pi. \end{cases}$ Plot $S_{10}(x)$ for $a = 3, b = 1$.

24. $f(x) = \begin{cases} x+1, & -1 < x < 0 \\ x-1, & 0 < x < 1. \end{cases}$ Plot $S_{10}(x)$.

25. $f(x) = 1 - |x|, \quad -1 < x < 1$. Plot $S_{10}(x)$.

26. $f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 \le x < 2. \end{cases}$ Plot $S_8(x)$.

27. $f(x) = |\sin x|, -\pi \le x \le \pi$ (a fully rectified sine wave). Plot $S_{10}(x)$.

28. $f(x) = \begin{cases} ax, & -\pi < x \le 0 \\ bx, & 0 \le x < \pi. \end{cases}$ Plot $S_8(x)$ for $a = 1, b = 3$.

29. $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi. \end{cases}$ Plot $S_8(x)$.

30. $f(x) = x^2, \quad -\pi \le x \le \pi$. Plot $S_8(x)$.

31. $f(x) = x^2, \quad -2\pi \le x \le 2\pi$. Plot $S_{10}(x)$.

32. $f(x) = \sin ax, \quad -\pi \le x \le \pi$ with a not an integer. Plot $S_{10}(x)$ for $a = 0.7$.

33. $f(x) = \cos ax, \quad -\pi \le x \le \pi$ with a not an integer. Plot $S_{10}(x)$ for $a = 0.7$.

34. $f(x) = e^{ax}, \quad -\pi \le x \le \pi$. Plot $S_7(x)$ for $a = 0.7$.

35. $f(x) = \begin{cases} 0, & -2\pi \le x < -\pi \\ \sin x, & -\pi \le x \le \pi \\ 0, & \pi \le x \le 2\pi. \end{cases}$ Plot $S_8(x)$.

9.2 Convergence of Fourier Series and Their Integration and Differentiation

The general theory of the convergence of Fourier series is complicated and still incomplete in some respects. Consequently, we will only derive some useful results that can be obtained in a straightforward manner, and then state without proof a convergence theorem due to the German mathematician P. G. L. Dirichlet (1805–1859) that is sufficient for all practical applications of Fourier series.

Let us consider the nth partial sum

$$S_n(x) = a_0 + \sum_{r=1}^{n} (a_r \cos rx + b_r \sin rx), \tag{20}$$

of the Fourier series for $f(x)$ in (7) defined over the interval $-\pi \le x \le \pi$. Then, provided the integral $\int_{-\pi}^{\pi} [f(x)]^2 dx$ exists and is finite, we have the obvious result

$$\int_{-\pi}^{\pi} [f(x) - S_n(x)]^2 dx = \int_{-\pi}^{\pi} [f(x)]^2 dx - 2\int_{-\pi}^{\pi} f(x)S_n(x)dx + \int_{-\pi}^{\pi} [S_n(x)]^2 dx. \tag{21}$$

From the definition of $S_n(x)$ in (20), it follows that

$$\int_{-\pi}^{\pi} [S_n(x)]^2 dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{r=1}^{n} (a_r \cos rx + b_r \sin rx) \right]^2 dx,$$

but the orthogonality of the sine and cosine functions reduces this to

$$\begin{aligned} \int_{-\pi}^{\pi} [S_n(x)]^2 dx &= \int_{-\pi}^{\pi} a_0^2 dx + \sum_{r=1}^{n} \left[a_r^2 \int_{-\pi}^{\pi} \cos^2 rx dx \right] + \sum_{r=1}^{n} \left[b_r^2 \int_{-\pi}^{\pi} \sin^2 rx dx \right] \\ &= \pi \left[2a_0^2 + \sum_{r=1}^{n} (a_r^2 + b_r^2) \right]. \end{aligned} \tag{22}$$

If $f(x)$ is replaced by its Fourier series, a similar argument shows that

$$\int_{-\pi}^{\pi} f(x)S_n(x)dx = \pi \left[2a_0^2 + \sum_{r=1}^{n} (a_r^2 + b_r^2) \right], \tag{23}$$

so combining (21) to (23) gives

$$\int_{-\pi}^{\pi} [f(x) - S_n(x)]^2 dx = \int_{-\pi}^{\pi} [f(x)]^2 dx - \pi \left[2a_0^2 + \sum_{r=1}^{n} (a_r^2 + b_r^2) \right]. \tag{24}$$

The integral on the left of (24) is nonnegative, because its integrand is a squared quantity, so it follows at once that for all n

$$2a_0^2 + \sum_{r=1}^{n}(a_r^2 + b_r^2) \le \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx,$$

so letting $n \to \infty$ we arrive at the inequality

$$2a_0^2 + \sum_{r=1}^{\infty}(a_r^2 + b_r^2) \le \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx. \qquad (25)$$

Bessel's inequality

This is **Bessel's inequality** for Fourier series, and the restriction to functions $f(x)$ such that $\int_{-\pi}^{\pi}[f(x)]^2 dx$ exists and is finite implies that the series

$$2a_0^2 + \sum_{r=1}^{\infty}(a_r^2 + b_r^2)$$

is convergent, so the coefficients in the associated Fourier series (7) must be such that

$$\lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \lim_{n\to\infty} b_n = 0. \qquad (26)$$

the fundamental Riemann–Lebesgue lemma

This important result on the behavior of Fourier coefficients as $n \to \infty$ is called the **Riemann–Lebesgue** lemma, though its rigorous proof proceeds differently.

It is also a consequence of (24) that if the nth partial sum $S_n(x)$ converges to $f(x)$ in the sense that

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}[f(x) - S_n(x)]^2 dx = 0,$$

which is true for all functions $f(x)$ encountered in applications, then

$$2a_0^2 + \sum_{r=1}^{\infty}(a_r^2 + b_r^2) = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx. \qquad (27)$$

Parseval relation

This is the **Parseval relation** for Fourier series.

EXAMPLE 9.6

Apply the Parseval relation to the Fourier series of $f(x) = |x|$ defined over the interval $-\pi \le x \le \pi$.

Solution It follows from Example 9.4 with $L = \pi$ that the Fourier series representation of $f(x) = |x|$ over the interval $-\pi \le x \le \pi$ is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2},$$

so that

$$a_0 = \frac{\pi}{2}, \quad a_{2n-1} = -\frac{4}{\pi(2n-1)^2}, \quad \text{and} \quad a_{2n} = 0 \quad \text{for } n = 1, 2, \ldots.$$

We have

$$\int_{-\pi}^{\pi}[f(x)]^2 dx = \int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^3}{3},$$

so as the integral is finite, provided $S_n(x)$ converges in the norm to $f(x)$, it follows from the Parseval relation in (27) that

$$\frac{1}{\pi}\left(\frac{2\pi^3}{3}\right) = 2\frac{\pi^2}{4} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}.$$

After simplification this reduces to the well-known result

$$\frac{\pi^4}{96} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$$
$$= \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots.$$

The justification for applying the Parseval relation in this case is provided by the following theorem. It can be confirmed by summing a large number of terms and comparing the result with the known value of $\pi^4/96$. For example, using $n = 100$ leads to the result $\pi^4/96 \approx 1.01467801$, while a direct calculation shows that $\pi^4/96 = 1.01467803$, so the two results agree to seven decimal places. ■

THEOREM 9.1

Convergence of Fourier series Let $f(x)$ be continuous over the interval $-L < x < L$ except possibly at a finite number of internal points $x_1, x_2, \ldots,$ at each point x_n of which the function has a finite jump discontinuity $f(x_n+) - f(x_n-)$. Furthermore, let the left- and right-hand derivatives $f'(x_n-)$ and $f'(x_n+)$ exist for $n = 1, 2, \ldots$. Then at points of continuity of $f(x)$ its Fourier series converges uniformly to $f(x)$, and at each point of discontinuity it converges pointwise to

fundamental convergence theorem

$$\frac{1}{2}(f(x_n-) + f(x_n+)) \quad \text{for } n = 1, 2, \ldots.$$

If, in addition, $f(x)$ has a right-hand derivative $f'(-L+)$ at the left end point of the interval and a left-hand derivative $f'(L-)$ at the right end point of the interval, then at $x = \pm L$ the Fourier series converges pointwise to

$$\frac{1}{2}(f(-L+) + f(L-)).$$

■

In effect, this theorem says that if $f(x)$ is piecewise continuous and bounded over the interval $-L < x < L$ with derivatives defined to the left and right of each discontinuity, its Fourier series converges uniformly to $f(x)$ wherever it is continuous and to the mid-point of the jump where there is a discontinuity. If, in addition, one-sided derivatives exist at the ends of the interval, then at both $x = -L$ and $x = L$ the Fourier series converges to the average of the values of $f(x)$ at the two ends of the interval.

A consequence of this theorem that is sometimes useful is that it allows many numerical series to be summed in closed form. Results of this type follow by choosing a value of x for which the terms of the Fourier series take on a simple numerical form, and equating the result to the appropriate value of $f(x)$. At a point $x = x^*$ where $f(x)$ is continuous the series will converge to $f(x^*)$, and at a point $x = x^*$ where $f(x)$ is discontinuous the series will converge to the mid-point of the jump.

EXAMPLE 9.7

(a) Given that the step function

$$f(x) = \begin{cases} -1, & \text{for } -\pi < x < 0 \\ 1, & \text{for } 0 < x < \pi \end{cases}$$

has the Fourier series

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1},$$

find a series for $\pi/4$.

(b) Given that

$$f(x) = \begin{cases} 0, & \text{for } -\pi < x < 0 \\ x^2, & \text{for } 0 \le x < \pi \end{cases}$$

has the Fourier series

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left\{\left[\frac{2(-1)^n}{n^2}\right]\cos nx + \frac{1}{\pi}\left[(-1)^n\left(\frac{2}{n^3} - \frac{\pi^2}{n}\right) - \frac{2}{n^3}\right]\sin nx\right\},$$

find a series for $\pi^2/6$.

Solution

(a) The function $f(x)$ graphed in Fig. 9.9 is seen to be discontinuous at $x = 0$ and to have different values at $x = \pm\pi$. The average of the values of $f(x)$ to the immediate left and right of the discontinuity at $x = 0$ is zero, so the Fourier series will converge to the value zero when $x = 0$. Setting $x = 0$ in the Fourier series causes every term to vanish, so equating this to the value to which the Fourier series converges at the origin yields the uninteresting result $0 = 0$.

how Fourier series can be used to sum series

To obtain a more interesting result, let us try setting $x = \pi/2$, which makes $\sin(2n-1)\frac{\pi}{2} = (-1)^{n+1}$. The function $f(x)$ is continuous at this point and equal to 1, so its Fourier series will converge to the value 1 when $x = \pi/2$. Inserting this value of x into the Fourier series and equating the result to 1 gives

$$1 = \frac{4}{\pi}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \cdots\right),$$

so

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}.$$

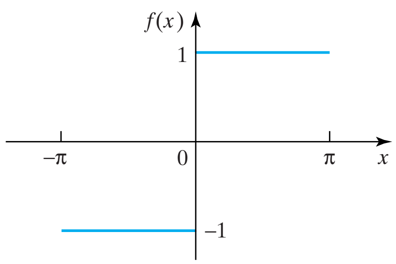


FIGURE 9.9 The step function $f(x)$.

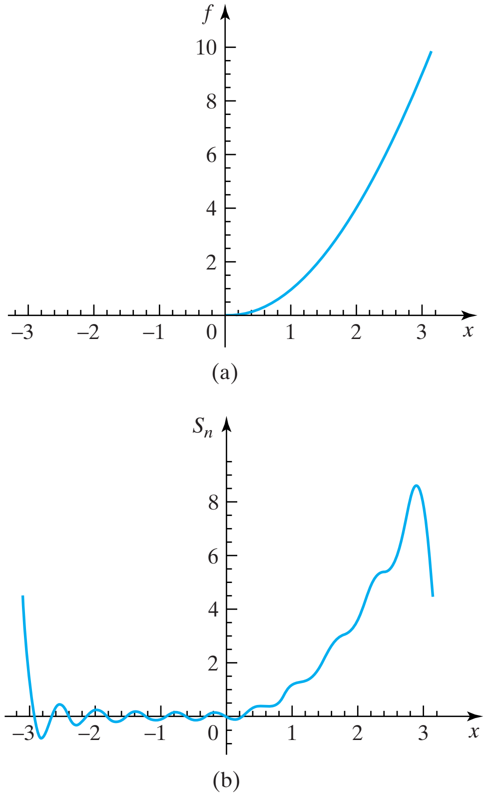


FIGURE 9.10 (a) The function $f(x)$ and (b) $S_{10}(x)$.

This series, known as Leibniz' formula, converges very slowly, so it is not useful for computing π.

(b) The function $f(x)$ is graphed in Fig. 9.10(a), and $S_{10}(x)$ in Fig. 9.10(b). The average of the values of $f(x)$ at the end points of the interval $-\pi < x < \pi$ is $\pi^2/2$, so setting $x = \pi$ in the Fourier series and equating the result to $\pi^2/2$ as required by the last part of Theorem 9.2 gives

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{1}{n^2},$$

where we have used the fact that $\cos n\pi = (-1)^n$ and $\sin n\pi = 0$ for positive integers n.

This result simplifies to the series

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty}\frac{1}{n^2},$$

which converges somewhat faster than the series in part (a). ■

Gibbs phenomenon

Examination of Fig. 9.3 and also Fig. 9.6 in Section 9.1 shows that when $f(x)$ is discontinuous, the graph of the partial sum $S_n(x)$ of the Fourier series representation of the function exhibits over- and undershoots close to the discontinuities. This is called the **Gibbs phenomenon**, and it persists for all values of n. This behavior

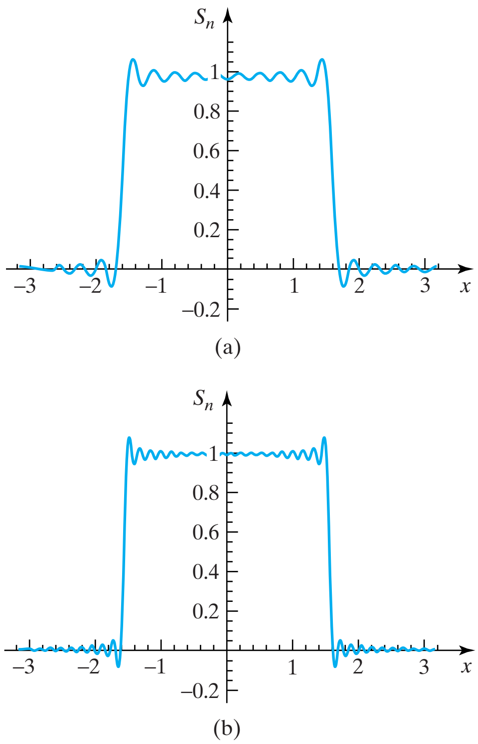


FIGURE 9.11 An example of the Gibbs phenomenon with (a) $n = 10$, and (b) $n = 20$.

reflects the way the continuous function $S_n(x)$ obtained from the Fourier series approximates the behavior of $f(x)$ at a point of discontinuity. Increasing n simply moves the under- and overshoots closer to the discontinuity while leaving their size approximately the same.

Figure 9.11 shows the Gibbs phenomena for the function

$$f(x) = \begin{cases} 0, & -\pi < x < -\pi/2 \\ 1, & -\pi/2 < x < \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$$

for different partial sums $S_n(x)$. The results should be compared with Fig. 9.3, which shows the graph of $S_5(x)$.

We now state without proof two important theorems concerning the term-by-term integration and differentiation of Fourier series that are often useful, but before doing so we first define what are called Dirichlet conditions, which are satisfied by most functions of practical importance.

A function $f(x)$ is said to satisfy **Dirichlet conditions** on an interval $-L < x < L$ if it is bounded on the interval, has at most a finite number of maxima and minima, and is continuous apart from a finite number of discontinuities in the interval.

THEOREM 9.2

when a Fourier series can be integrated

Termwise integration of Fourier series The integral of any function $f(x)$ satisfying Dirichlet conditions on the interval $-L \le x \le L$ can be obtained by term-by-term integration of the Fourier series representation of $f(x)$. So, if $f(x)$ has the Fourier

series representation

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) \quad \text{for } -L \le x \le L,$$

then

$$\int_{-L}^{x} f(u)du = a_0(x+L) + \frac{L}{\pi}\sum_{n=1}^{\infty}\left[\frac{a_n}{n}\sin\left(\frac{n\pi x}{L}\right) - \frac{b_n}{n}\left(\cos\left(\frac{n\pi x}{L}\right) + (-1)^{n+1}\right)\right] \quad \text{for } -L \le x \le L.$$

■

THEOREM 9.3

when a Fourier series can be differentiated

Termwise differentiation of Fourier series Let $f(x)$ be a continuous function on the interval $-L \le x \le L$ such that $f(-L) = f(L)$, and suppose also that $f'(x)$ is piecewise continuous. Then for any x strictly inside the interval at which $f''(x)$ exists, the derivative of $f(x)$ can be obtained by term-by-term differentiation of the Fourier series representation of $f(x)$. So, if $f(x)$ has the Fourier series representation

$$f(x) = a_0 + \frac{\pi}{L}\sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) \quad \text{for } -L \le x \le L,$$

then

$$f'(x) = \frac{\pi}{L}\sum_{n=1}^{\infty}\left(-na_n \sin\left(\frac{n\pi x}{L}\right) + nb_n \cos\left(\frac{n\pi x}{L}\right)\right) \quad \text{for } -L < x < L,$$

except for points at where $f'(x)$ and $f''(x)$ are not defined. ■

EXAMPLE 9.8

Use the Fourier series representation of the function

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

given in Example 9.7 to find a Fourier series representation of $F(x) = \int_{-\pi}^{x} f(t)dt$ in the interval $-\pi < x < \pi$, and relate the result to Example 9.4.

Solution As $f(x)$ satisfies the conditions of Theorem 9.2, its Fourier series representation may be integrated term by term to obtain the Fourier series representation of

$$F(x) = \int_{-\pi}^{x} f(t)dt = \begin{cases} \displaystyle\int_{-\pi}^{x} -1dt = -(x+\pi), & \text{for } -\pi < x < 0 \\ \displaystyle\int_{-\pi}^{0} -1dt + \int_{0}^{x} 1dt = x - \pi & \text{for } 0 < x < \pi. \end{cases}$$

From Example 9.7, the Fourier series representation of $f(x)$ is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1},$$

so replacing x by the dummy variable t and integrating over the interval $-\pi \le t \le x$ gives

$$F(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{x}\frac{\sin(2n-1)t}{2n-1}dt = -\frac{4}{\pi}\left[\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - \sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi}{(2n-1)^2}\right].$$

As $\cos(2n-1)\pi = -1$ for $n = 1, 2, \ldots,$ this reduces to

$$F(x) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

The numerical series on the right can be summed by applying the Parseval relation to the Fourier series representation of $f(x)$ to obtain

$$2 = \sum_{n=1}^{\infty}\left(\frac{4}{\pi(2n-1)}\right)^2, \quad \text{or} \quad \frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

Replacing the numerical series in $F(x)$ by $\pi^2/8$ reduces it to

$$\int_{-\pi}^{x} f(t)dt = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - \frac{4}{\pi}\frac{\pi^2}{8} = -\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2},$$

and so the required Fourier series representation is

$$F(x) = \left\{\begin{array}{ll} \int_{-\pi}^{x} -1dt = -(x+\pi), & \text{for } -\pi < x < 0 \\ \int_{-\pi}^{0} -1dt + \int_{0}^{x} 1dt = x - \pi, & \text{for } 0 < x < \pi \end{array}\right\}$$

$$= -\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}.$$

Examination of $F(x)$ shows that $F(x) = |x| - \pi$, so as a check we see that the Fourier series representation of the function $|x|$ in the interval $-\pi \le x \le \pi$ can be obtained by adding π to the Fourier series representation of $F(x)$ to obtain

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}, \quad \text{for } -\pi \le x \le \pi,$$

in agreement with the result of Example 9.4 with $L = \pi$. ■

EXAMPLE 9.9 Given

$$f(x) = \begin{cases} \sin 2x, & -\pi \le x < -\pi/2 \\ 0, & -\pi/2 \le x \le \pi/2 \\ \sin 2x, & \pi/2 < x \le \pi, \end{cases}$$

find $f'(x)$ by differentiation of the Fourier series representation of $f(x)$.

Solution The function satisfies the conditions of Theorem 9.3, so its Fourier series representation may be differentiated term by term to find the Fourier series representation of $f'(x)$. It was shown in Example 9.2 that the Fourier series representation of $f(x)$ is

$$f(x) = \frac{1}{2\pi} + \frac{1}{\pi}\left(\frac{2}{3}\cos x - \frac{2}{5}\cos 3x - \frac{1}{3}\cos 4x - \cdots\right) + \frac{1}{\pi}\left(-\frac{2}{3}\sin x + \frac{3\pi}{4}\sin 2x - \frac{2}{5}\sin 3x + \cdots\right),$$

so differentiation shows the first few terms of the Fourier series for $f'(x)$ to be

$$f'(x) = \frac{1}{\pi}\left(-\frac{2}{3}\sin x + \frac{6}{5}\sin 3x + \cdots\right) + \frac{1}{\pi}\left(-\frac{2}{3}\cos x + \frac{3\pi}{2}\cos 2x - \cdots\right),$$

where from the definition of $f(x)$

$$f'(x) = \begin{cases} 2\cos 2x, & -\pi \le x < -\pi/2 \\ 0, & -\pi/2 \le x \le \pi/2 \\ 2\cos 2x, & \pi/2 < x \le \pi. \end{cases}$$ ■

Summary

The convergence of Fourier series has been examined, and it has been shown that where $f(x)$ is continuous its Fourier series representation converges to $f(x)$, but where it has a finite jump discontinuity it converges to the mid-point of the jump. The Bessel inequality and the Parseval relation have been established, and conditions given for the termwise integration and differentiation of a Fourier series.

EXERCISES 9.2

In Exercises 1 through 4, apply the Parseval relation to the given function and its Fourier series to obtain a series representation involving a power of π.

1. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$

with $f(x) = \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}$.

2. $f(x) = x,\ -\pi < x < \pi$

with $f(x) = 2\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$.

3. $f(x) = x^2,\ -\pi \le x \le \pi$,

with $f(x) = \dfrac{\pi^2}{3} - 4\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos nx}{n^2}$.

4. $f(x) = |\cos x|,\ -\pi \le x \le \pi$

with $f(x) = \dfrac{2}{\pi} + \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos 2nx}{(4n^2-1)}$.

5. Show that the **Parseval relation** for a function $f(x)$ defined on the interval $-L < x < L$ takes the form

$$\frac{1}{L}\int_{-L}^{L}[f(x)]^2 dx = 2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

6. Find the Fourier series for the function

$$f(x) = \begin{cases} 0, & -4 \le x < 0 \\ 4, & 0 \le x < 4 \end{cases}$$

and apply the Parseval relation in Exercise 5 to the result.

7. Use the Fourier series in Example 10.6(b) for the function

$$f(x) = \begin{cases} 0, & \text{for } -\pi \le x \le 0 \\ x^2, & \text{for } 0 < x < \pi \end{cases}$$

to find a series for $\pi^2/12$.

8. Use the Fourier series for $f(x) = |\sin x|$, for $-\pi \le x \le \pi$, to find a series for $\pi/4$.

9. Use the Fourier series for

$$f(x) = \begin{cases} 0, & \text{for } -1 < x < 0 \\ x, & \text{for } 0 \le x < 1 \end{cases}$$

to find a series for $\pi^2/8$.

10. Integrate the Fourier series of $f(x)$ in Exercise 2 to find the Fourier series of x^2. What happens if the Fourier series of $f(x)$ is differentiated to find $f'(x)$?

11. Find the Fourier series of $f(x) = \pi^2 - x^2$ for $-\pi \le x \le \pi$ and use it with Theorems 10.2 and 10.3 to find the Fourier series of x and $x(\pi^2 - x^2)$.

Exercises 12 through 18 are optional. Exercises 12 through 14 show how the partial sum

$$S_n(x) = a_0 + \sum_{r=1}^{n}(a_r \cos rx + b_r \sin rx),$$

of the Fourier series of a function $f(x)$ defined over the fundamental interval $-\pi \le x \le \pi$, and by periodic extension outside it, can be expressed as an integral. Exercises 15 through 17 provide an intuitive justification of Theorem 9.1.

12. Starting from the trigonometric identity

$$\frac{1}{2} + \sum_{r=1}^{n} \cos rx = \frac{\sin\left[\left(n+\frac{1}{2}\right)x\right]}{2\sin\left(\frac{x}{2}\right)}$$

that formed Exercise 19 in Section 1.4, integrate the identity first over the interval $[-\pi, 0]$ and then over the interval $[0, \pi]$ to show that

$$\int_{-\pi}^{0} \frac{\sin\left[\left(n+\frac{1}{2}\right)x\right]}{\sin\left(\frac{x}{2}\right)}dx = \pi \quad \text{and}$$

$$\int_{0}^{\pi} \frac{\sin\left[\left(n+\frac{1}{2}\right)x\right]}{\sin\left(\frac{x}{2}\right)}dx = \pi.$$

13. Substitute the Euler formulas for a_r and b_r into $S_n(x)$, after first replacing the dummy variable x in each integral by the dummy variable u to avoid confusion with the variable x in $S_n(x)$. Combine all terms under a single integral sign and, after simplifying the result using the formula $\cos a \cos b + \sin a \sin b = \cos(a - b)$, use the results of Exercise 12 to show that

$$S_n(x) = \frac{1}{\pi}\int_{x-\pi}^{x+\pi} f(x-t)\frac{\sin\left[\left(n+\frac{1}{2}\right)t\right]}{2\sin\left(\frac{t}{2}\right)}dt.$$

14. Use the periodicity of the integrand of $S_n(x)$ in Exercise 13 to show that

$$S_n(x) = \frac{1}{\pi}\int_{0}^{\pi} [f(x-t) + f(x+t)]\frac{\sin\left[\left(n+\frac{1}{2}\right)t\right]}{2\sin\left(\frac{t}{2}\right)}dt.$$

The function $D_n(t) = \sin[(n+\frac{1}{2})t]/[2\sin(\frac{t}{2})]$ occurring in the integrand of $S_n(x)$ is called the **Dirichlet kernel**.

15. Use a computer to graph $D_n(t)$ in Exercise 14 in the interval $-\pi \le t \le \pi$, for $n = 10, 15, 30$. Confirm from the graphs that when n is large $D_n(t)$ only differs significantly from zero in the interval $-2\pi/(2n+1) \le t \le 2\pi/(2n+1)$.

16. Use the conclusion of Exercise 15 together with the result

$$\int_{-\pi}^{\pi} D_n(t)dt = \pi$$

established in Exercise 12 to give reasons why for large n the Dirichlet kernel $D_n(t)$ can be approximated by the rectangular pulse function

$$\Delta(t) = \begin{cases} 0, & -\pi \le t < -2\pi(2n+1) \\ (2n+1)/4, & -2\pi/(2n+1) \le t \le 2\pi/(2n+1) \\ 0, & 2\pi/(2n+1) < t \le \pi. \end{cases}$$

17. Use the result of Exercise 16, with

$$S_n(x) = \frac{1}{\pi}\int_{0}^{\pi} [f(x-t) + f(x+t)]D_n(t)dt$$

from Exercise 14, to suggest why in the limit as $n \to \infty$ this confirms the convergence properties of Fourier series stated in Theorem 9.1.

18. By first setting $f(x) = \sin mx$ and then $f(x) = \cos mx$ in the result of Exercise 17, with m a positive integer, and using the fact that the functions $\sin mx$ and $\cos mx$ are their own Fourier series on $-\pi \le x \le \pi$, deduce that

$$\int_{0}^{\pi} \sin mt\, D_n(t)dt = \int_{0}^{\pi} \cos mt\, D_n(t)dt = \begin{cases} 0, & n = 1, 2, \ldots, m-1 \\ \pi/2, & n = m, m+1, \ldots. \end{cases}$$

9.3 Fourier Sine and Cosine Series on $0 \le x \le L$

A function $f(x)$ that is specified on the interval $0 \le x \le L$ can be represented in terms of a series either of sines or of cosines on the interval. These series are obtained by first extending the definition of the function to the interval $-L \le x \le L$ in a suitable manner, and then restricting the Fourier series representation of the extended function to the original interval $0 \le x \le L$.

Sine Series on $0 \leq x \leq L$

Let a function $f(x)$ specified on the interval $0 \leq x \leq L$ be extended to the interval $-L \leq x \leq L$ as an odd function by the requirement that $f(-x) = -f(x)$ for $-L \leq x \leq L$. Then the odd function $g(x)$ given by

$$g(x) = \begin{cases} -f(-x), & -L \leq x \leq 0 \\ f(x), & 0 \leq x \leq L, \end{cases}$$

and defined on the interval $-L \leq x \leq L$, coincides with the function $f(x)$ on the original interval $0 \leq x \leq L$.

It follows from Theorem 9.1 and the Fourier series representation of functions on the interval $-L \leq x \leq L$ that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \quad \text{for } -L \leq x \leq L, \tag{28}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \quad \text{for } n = 1, 2, \ldots. \tag{29}$$

As the functions $f(x)$ and $g(x)$ coincide for $0 \leq x \leq L$, we see that by restricting x to the interval $0 \leq x \leq L$, series (28) is the required sine series. Result (28) with the coefficients b_n defined by (29) is called the **sine series** representation of $f(x)$ on the interval $0 \leq x \leq L$, or sometimes the **half-range sine series expansion** of $f(x)$.

Cosine Series on $0 \leq x \leq L$

If $f(x)$ is extended to the interval $-L \leq x \leq L$ as an even function, by requiring that $f(-x) = f(x)$ for $-L \leq x \leq 0$, we can define an even function $g(x)$ by

$$g(x) = \begin{cases} f(-x), & -L \leq x \leq 0 \\ f(x), & 0 \leq x \leq L. \end{cases}$$

If we again use Theorem 9.1 with the Fourier series representation of functions on the interval $-L \leq x \leq L$, it follows that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad \text{for } -L \leq x \leq L \tag{30}$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)dx \quad \text{and} \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \quad \text{for } n = 1, 2, \ldots. \tag{31}$$

Here also the functions $f(x)$ and $g(x)$ coincide for $0 \leq x \leq L$, so by restricting x to this interval (30) is seen to provide required cosine series representation of $f(x)$ on the interval $0 \leq x \leq L$. Result (31) with the coefficients a_n defined by (32) is called the **cosine series** representation of $f(x)$ on the interval $0 \leq x \leq L$, or sometimes the **half-range cosine series expansion** of $f(x)$.

Fourier expansions only in terms of sines or cosines

Sine and cosine representations of $f(x)$ on $0 \le x \le L$

Let $f(x)$ be defined on the interval $0 \le x \le L$. Then the **sine series** representation of $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad \text{for } 0 \le x \le L,$$

where

$$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad \text{for } n = 1, 2, \dots,$$

and the **cosine series** representation of $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad \text{for } 0 \le x \le L,$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x) dx \quad \text{and} \quad a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L} dx,$$

$$\text{for } n = 1, 2, \dots.$$

EXAMPLE 9.10 Find the sine and cosine representations of $f(x) = x$ for $0 \le x \le \pi$.

Solution The sine series representation is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx dx, \quad \text{for } n = 1, 2, \dots.$$

Integrating this last result, we find that

$$b_n = (-1)^{n+1}\frac{2}{n},$$

so the required sine series representation is

$$f(x) = 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n} \quad \text{for } 0 \le x \le \pi.$$

The cosine series representation is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

where

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x dx \quad \text{and} \quad a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx dx \quad \text{for } n = 1, 2, \ldots.$$

Integration gives

$$a_0 = \frac{\pi}{2}, \quad \text{while } a_{2n-1} = -\frac{4}{\pi(2n-1)^2}, \quad \text{and} \quad a_{2n} = 0 \quad \text{for } n = 1, 2, \ldots,$$

so the cosine series representation is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} \quad \text{for } 0 \le x \le \pi.$$

■

Summary

It has been shown how a function $f(x)$ defined on the interval $0 \le x \le L$ can be represented either in terms of a series involving only sine functions or as a series involving only cosine functions. These special Fourier series, called either half-range sine or cosine Fourier series, were obtained from the usual expansion over the interval $-L \le x \le L$ by extending the definition of $f(x)$ to the interval $-L \le x \le L$ in a suitable manner. As half-range Fourier series are derived from ordinary Fourier series, their convergence properties are the same as those of ordinary Fourier series.

EXERCISES 9.3

In Exercises 1 through 4 find the sine series for the given function defined on the interval $0 \le x \le \pi$.

1. $f(x) = x^2$.

2. $f(x) = |\cos x|$.

3. $f(x) = \begin{cases} \cos x, & 0 < x \le \pi/2 \\ 0, & \pi/2 < x \le \pi. \end{cases}$

4. $f(x) = (x - \pi)^2/\pi^2$.

In Exercises 5 through 8 find the cosine series for the given function defined on the interval $0 \le x \le \pi$.

5. $f(x) = \begin{cases} \cos x, & 0 < x \le \pi/2 \\ 0, & \pi/2 < x \le \pi. \end{cases}$

6. $f(x) = \sin x$.

7. $f(x) = \begin{cases} \sin x, & 0 < x \le \pi/2 \\ 0, & \pi/2 < x \le \pi. \end{cases}$

8. $f(x) = (x - \pi)^2/\pi^2$.

9. Use the sine series together with the orthogonality of the functions $\sin \frac{n\pi x}{L}$, for $n = 1, 2, \ldots$, on the interval $0 \le x \le L$ to show that the **Parseval relation** for the **sine series** takes the form

$$\frac{2}{L}\int_0^L [f(x)]^2 dx = \sum_{n=1}^{\infty} b_n^2.$$

10. Use the cosine series together with the orthogonality of the functions $\cos \frac{n\pi x}{L}$, for $n = 1, 2, \ldots$, on the interval $0 \le x \le L$ to show that the **Parseval relation** for the **cosine series** takes the form

$$\frac{2}{L}\int_0^L [f(x)]^2 dx = 2a_0^2 + 0^2 + \sum_{n=1}^{\infty} a_n^2.$$

11. Find the sine series representation of

$$f(x) = e^{-x}, \quad 0 < x < \pi.$$

12. Find the sine and cosine series representations of $f(x) = \pi - x$ on the interval $0 \le x \le \pi$. Use them with the results of Exercises 9 and 10 to show that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} \quad \text{and} \quad \frac{\pi^4}{96} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}.$$

Comment on which series representation converges most rapidly to $f(x)$.

13.* Explain why if $f(x)$ and $g(x)$ have Fourier series representations for $-\pi \le x \le \pi$, the Fourier series representations of $f(x) \pm g(x)$ can be obtained from those for $f(x)$ and $g(x)$ by term-by-term addition or subtraction. By adding and subtracting the Fourier series representations of

$$\int_{-\pi}^{\pi} [f(x) + g(x)]dx \quad \text{and} \quad \int_{-\pi}^{\pi} [f(x) - g(x)]dx,$$

obtain the **generalized Parseval relation**

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx = 2a_0 A_0 + \sum_{n=1}^{\infty}(a_n A_n + b_n B_n),$$

where the a_n, b_n are the Fourier coefficients of $f(x)$ and the A_n, B_n are the Fourier coefficients of $g(x)$.

14.* Let $f(x)$ defined for $-\pi \leq x \leq \pi$ be approximated by the nth partial sum of its Fourier series representation

$$S_n(x) = a_0 + \sum_{m=1}^{n}(a_m \cos mx + b_m \sin mx),$$

and let

$$\Phi(x) = A_0 + \sum_{m=1}^{n}(A_m \cos mx + B_m \sin mx)$$

be any other approximation to $f(x)$ with coefficients A_m and B_m. Show by expanding the square error

$$E_n = \int_{-\pi}^{\pi}[f(x) - \Phi_n(x)]^2 dx$$

in terms of the Fourier series representation of $f(x)$ that E_n is minimized when $A_m = a_m$ and $B_m = b_m$ for $m = 0, 1, 2, \ldots, n$. This establishes the fact that the Fourier series partial sum $S_n(x)$ provides the best trigonometric approximation to $f(x)$ in the least squares sense.

9.4 Other Forms of Fourier Series

In this section we introduce two other forms of Fourier series that prove useful. The first is the Fourier series of a function $f(x)$ defined over an interval $a - L \leq x \leq a + L$ with a an arbitrary real number, and by periodicity outside it. Frequently $a = L$, corresponding to the Fourier series over the interval $0 \leq x \leq 2L$. The second form of Fourier series considered uses the Euler identity $e^{ix} = \cos x + i \sin x$ to derive the **complex** form of the Fourier series, also often called the **exponential form** of the Fourier series.

Fourier Series over a Shifted Interval

Routine integration shows the set of functions

$$1, \quad \sin\frac{n\pi x}{L} \quad \text{and} \quad \cos\frac{n\pi x}{L} \quad \text{for } n = 1, 2, \ldots$$

form an orthogonal system over any interval of the form $a - L \leq x \leq a + L$, for any real number a, and that

$$\int_{a-L}^{a+L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = 0 \quad \text{for all integers } m \text{ and } n,$$

$$\int_{a-L}^{a+L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = \begin{cases} 0 & \text{for } m \neq n \\ L & \text{for } m = n, \text{ for all integers } m \text{ and } n, \end{cases}$$

$$\int_{a-L}^{a+L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = \begin{cases} 0 & \text{for } m \neq n \\ L & \text{for } m = n \neq 0 \\ 2L & \text{for } m = n = 0, \text{ for all integers } m \text{ and } n. \end{cases}$$

The following result is a direct consequence of these integrals, and it provides an extension of the definition of a Fourier series to the interval $-L \leq x \leq L$.

Fourier series over a shifted interval

Fourier series over the interval $a - L \leq x \leq a + L$

A function $f(x)$ defined on the interval $a - L \leq x \leq a + L$ has the Fourier series representation

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right), \tag{32}$$

where

$$a_0 = \frac{1}{2L}\int_{a-L}^{a+L} f(x)dx, \quad a_n = \frac{1}{L}\int_{a-L}^{a+L} f(x)\cos\frac{n\pi x}{L}dx,$$
$$b_n = \frac{1}{L}\int_{a-L}^{a+L} f(x)\sin\frac{n\pi x}{L}dx, \quad \text{for } n = 1, 2, \ldots. \tag{33}$$

EXAMPLE 9.11 Find the Fourier series representation of

$$f(x) = \begin{cases} x, & 0 \le x \le \pi \\ \pi, & \pi \le x < 2\pi. \end{cases}$$

Solution A graph of the function $f(x)$ is shown in Fig. 9.12. Using (33) with $a = L = \pi$ gives

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{3\pi}{4} \quad \text{and} \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx,$$

from which it follows that

$$a_{2n-1} = -\frac{2}{\pi(2n-1)^2} \quad \text{and} \quad a_{2n} = 0 \quad \text{for } n = 1, 2, \ldots.$$

The Euler formula for b_n gives

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx = -\frac{1}{n} \quad \text{for } n = 1, 2, \ldots,$$

so the required Fourier series is

$$f(x) = \frac{3\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - \sum_{n=1}^{\infty}\frac{\sin nx}{n} \quad \text{for } 0 \le x < 2\pi.$$ ■

Complex Fourier Series

The Euler identities $e^{ix} = \cos x + i\sin x$ and $e^{-ix} = \cos x - i\sin x$ allow us to write

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

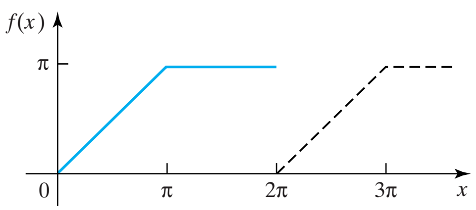


FIGURE 9.12 The function $f(x)$ defined for $0 \le x < 2\pi$.

When these results are used in the real variable Fourier series representation of $f(x)$ over the interval $-L \le x \le L$, it becomes

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{in\pi x/L} + e^{-in\pi x/L}}{2}\right) + b_n\left(\frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i}\right)\right],$$

and after grouping terms we have

$$f(x) = a_0 + \sum_{n=1}^{n}\left(\frac{a_n - ib_n}{2}\right)e^{in\pi x/L} + \sum_{n=1}^{n}\left(\frac{a_n + ib_n}{2}\right)e^{-in\pi x/L}. \tag{34}$$

If we now define

$$c_0 = a_0, \quad c_n = \frac{a_n - ib_n}{2}, \quad \text{and} \quad c_{-n} = \frac{a_n + ib_n}{2} \quad \text{for } n = 1, 2, \ldots, \tag{35}$$

the Fourier series representation of $f(x)$ in (34) becomes

$$f(x) = \lim_{k\to\infty}\sum_{n=-k}^{k} c_n e^{in\pi x/L} \quad \text{for } -L \le x \le L. \tag{36}$$

This is the **complex** or **exponential** form of the Fourier series representation of $f(x)$.

If real functions $f(x)$ are considered, the Fourier coefficients a_n and b_n are real, and (35) then shows that c_n and c_{-n} are complex conjugates, because $c_{-n} = \bar{c}_n$. To proceed further we now make use of the fact that the functions $\exp(im\pi x/L)$ and $\exp(-in\pi x/L)$ are orthogonal over the interval $-L \le x \le L$, because integration shows that

$$\int_{-L}^{L} e^{im\pi x/L}e^{-in\pi x/L}dx = \begin{cases} 0, & \text{for } m \ne -n \\ 2\pi & \text{for } m = -n \text{ for } m, n \text{ positive integers.}\end{cases}$$

Multiplication of (36) by $\exp(-im\pi x/L)$, followed by integration over $-L \le x \le L$ and use of the above orthogonality condition gives

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots. \tag{37}$$

Collecting these results we arrive at the following definition.

the complex or exponential form of a Fourier series

The complex form of a Fourier series

Let the real function $f(x)$ be defined on the interval $-L \le x \le L$. Then the complex Fourier series representation of $f(x)$ is

$$f(x) = \lim_{k\to\infty}\sum_{n=-k}^{k} c_n e^{in\pi x/L} \quad \text{for } -L \le x \le L,$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots.$$

As the complex form of a Fourier series was derived directly from the real variable Fourier series, it follows directly that if $f(x)$ is defined for $a - L \leq x \leq a + L$, then

$$f(x) = \lim_{k\to\infty} \sum_{n=-k}^{k} c_n e^{in\pi x/L} \quad \text{for } a - L \leq x \leq a + L, \tag{38}$$

with

$$c_n = \frac{1}{2L}\int_{a-L}^{a+L} f(x)e^{-in\pi x/L}dx, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots. \tag{39}$$

It is sometimes useful to separate out the coefficient c_0 from the summation in (36) (or in (38)) by writing

$$f(x) = c_0 + \lim_{k\to\infty} \sum_{n=-k}^{k}{}' c_n e^{in\pi x/L}, \tag{40}$$

with the understanding that Σ' indicates that the term corresponding to $n = 0$ has been omitted from the summation.

When $f(x)$ is real, so that $c_{-n} = \bar{c}_n$, result (40) becomes

$$f(x) = c_0 + \sum_{n=1}^{\infty}[c_n e^{in\pi x/L} + \bar{c}_n e^{-in\pi x/L}]. \tag{41}$$

Because the complex form of the Fourier series representation of a function is derived from its real variable definition, the convergence properties of complex Fourier series are the same as those already discussed for the real variable case. So at points of continuity of $f(x)$ the complex Fourier series converges uniformly to $f(x)$, while at points of discontinuity it converges to the mid-point of the jump discontinuity.

EXAMPLE 9.12 Find the complex Fourier series representation of

$$f(x) = \begin{cases} 0, & -\pi < x < -\pi/2 \\ 1, & -\pi/2 < x < \pi/2 \\ 0, & \pi/2 < x < \pi. \end{cases}$$

Solution As the function $f(x)$ is defined on the interval $-\pi \leq x \leq \pi$, we have $L = \pi$, so the coefficients c_n are given by

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1dx = \frac{1}{2}$$

and

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-inx}dx = \frac{1}{n\pi}\left(\frac{e^{in\pi/2} - e^{-in\pi/2}}{2i}\right)$$
$$\text{for } n = \pm 1, \pm 2, \ldots.$$

The coefficients c_n reduce to the real values

$$c_n = \frac{1}{n\pi}\sin\frac{n\pi}{2} \quad \text{for } n = \pm 1, \pm 2, \ldots,$$

so $c_n = c_{-n}$ because c_n is an even function of n. Consideration of the function

$\sin(n\pi/2)$ for integer values of n shows that

$$c_{2n-1} = \frac{(-1)^{n-1}}{\pi(2n-1)} \quad \text{and} \quad c_{2n} = 0 \quad \text{for } n = 1, 2, \ldots.$$

Thus, the complex Fourier series representation of $f(x)$ is

$$f(x) = \frac{1}{2} + \lim_{k\to\infty} {\sum_{n=-k}^{k}}' c_n(e^{inx} + e^{-inx}).$$

The real variable Fourier series representation of this function $f(x)$ was derived in Chapter 8, Example 8.22, and considered again at the start of Section 9.1. If c_n is used in the preceding result with $e^{inx} + e^{-inx} = 2\cos nx$, the complex Fourier series representation reduces to the real variable Fourier series representation

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(2n-1)x}{(2n-1)}$$

that was obtained previously. This series, and the equivalent complex series, converges uniformly to $f(x)$ at points of continuity of $f(x)$ and to the value 1/2 at the discontinuities located at $x = \pm\pi/2$. ■

EXAMPLE 9.13 Find the complex Fourier series representation of

$$f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 < x < 4. \end{cases}$$

Solution The function $f(x)$ is defined on the interval $0 \le x \le 2L$, with $2L = 4$, so $L = 2$. Thus, the complex Fourier coefficients c_n are given by

$$c_n = \frac{1}{4}\int_0^4 f(x)e^{-in\pi x/2}dx = \frac{1}{4}\int_1^4 e^{-in\pi x/2}dx, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots.$$

Setting $n = 0$ gives

$$c_0 = \frac{3}{4},$$

whereas

$$c_n = \frac{i}{2\pi n}\left[1 - e^{-in\pi/2}\right], \quad \text{for } n = \pm 1, \pm 2, \ldots.$$

So the complex Fourier series representation of $f(x)$ is

$$f(x) = c_0 + \lim_{k\to\infty}\sum_{n=-k}^{k} c_n e^{in\pi x/2},$$

with c_0 and c_n defined as shown. ■

Accounts of Fourier series and their general properties are to be found in references [3.3] to [3.5] and also in [3.7], [3.16], and [4.2]. An advanced and encyclopedic account of trigonometric series is given in reference [4.5].

Summary

Other forms of Fourier series have been derived, first by stretching and shifting the interval over which the expansion was required, and then by expressing the series in complex form. As both results were derived from the ordinary Fourier series, their convergence properties are the same as those of ordinary Fourier series.

EXERCISES 9.4

In Exercises 1 through 4 find the Fourier series representation of the function $f(x)$ over the given shifted interval.

1. $f(x) = \begin{cases} 0, & 0 < x < \pi \\ 1, & \pi < x < 2\pi. \end{cases}$
2. $f(x) = 1 - x, \quad 0 < x < 1.$
3. $f(x) = x, \quad 0 < x < \pi.$
4. $f(x) = x^2, \quad \pi < x < 3\pi.$

In Exercises 5 through 10 find the complex Fourier series representations of the given function $f(x)$ over the stated interval.

5. $f(x) = e^x, \quad -1 < x < 1.$
6. $f(x) = x^2, \quad 0 < x < 2\pi.$
7. $f(x) = e^x, \quad 0 < x < 1.$
8. $f(x) = \sinh x, \quad -\pi < x < \pi.$
9. $f(x) = e^x, \quad -\pi < x < \pi.$
10. $f(x) = \cosh x, \quad -1 < x < 1.$

9.5 Frequency and Amplitude Spectra of a Function

When Fourier series are applied to periodic physical phenomena with period T, it is convenient to work in terms of the angular frequency ω_0 defined as

$$\omega_0 = \frac{2\pi}{T}, \tag{42}$$

where $1/T = \omega_0/2\pi$ measures the number of cycles (oscillations) occurring in one time unit. For example, the period of the function $\sin 2x$ is $T = \pi$, so in this case $\omega_0 = 2$.

interpreting Fourier series representations in a different way

The Fourier series representation of a function $f(x)$ defined on the interval $-L \leq x \leq L$ with the corresponding period $T = 2L$ has been shown to be

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right),$$

so as $\omega_0 = \pi/L$ this can be written

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 x + b_n \sin n\omega_0 x), \tag{43}$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^{L} f(x)\cos n\omega_0 x dx \quad \text{for } n = 1, 2, \ldots, \tag{44}$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^{L} f(x)\sin n\omega_0 x dx \quad \text{for } n = 1, 2, \ldots. \tag{45}$$

In terms of these results (43) becomes

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)^{1/2} \left[\frac{a_n}{(a_n^2 + b_n^2)^{1/2}} \cos n\omega_0 x + \frac{b_n}{(a_n^2 + b_n^2)^{1/2}} \sin n\omega_0 x \right]. \tag{46}$$

Using the trigonometric identity $\cos(P + Q) = \cos P \cos Q - \sin P \sin Q$, and defining

$$A_n = (a_n^2 + b_n^2)^{1/2} \quad \text{and} \quad \delta_n = \text{Arctan}\,(-b_n/a_n), \tag{47}$$

with A_n the **amplitude** and δ_n the **phase**, allows (46) to be written more concisely in the **amplitude and phase angle** representation

$$f(x) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 x + \delta_n). \tag{48}$$

When the Fourier series representation of $f(x)$ is expressed in this form, the set of numbers

$$\omega_0, \quad 2\omega_0, \quad 3\omega_0, \ldots$$

frequency spectrum, amplitude, and phase

is called the **frequency spectrum** of the function $f(x)$. The number $n\omega_0$ is called the ***n*th harmonic frequency** of $f(x)$, and the number δ_n the ***n*th phase angle** of $f(x)$. The set of numbers

$$A_0, \quad A_1, \quad A_2, \ldots,$$

where $A_0 = |a_0|$, is called the **amplitude spectrum** of $f(x)$, and the function

$$\cos(n\omega_0 x + \delta_n)$$

is called the ***n*th harmonic** of the function $f(x)$. The amplitude spectrum can be displayed graphically by drawing lines of height $A_0, A_1, A_2, \ldots,$ against the respective harmonic frequencies $\omega_0, 2\omega_0, 3\omega_0, \ldots,$ as shown in the next example. This is called a **discrete spectrum**, because the amplitude is only defined at the discrete frequencies in the frequency spectrum.

Result (48) shows how $f(x)$ is representable in terms of a linear combination of harmonics, each weighted by an appropriate amplitude factor A_n.

EXAMPLE 9.14 Find the harmonics and amplitude spectrum of

$$f(x) = \begin{cases} \pi, & -\pi < x < 0 \\ \pi - x, & 0 \le x \le \pi. \end{cases}$$

Solution In this case the function is defined on the interval $-\pi \le x \le \pi$, so $L = \pi$, $T = 2L = 2\pi$, and $\omega_0 = 2\pi/T = 1$. The frequency spectrum becomes $1, 2, 3, \ldots,$

and the Fourier series representation in terms of frequency is

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{0} \pi\, dx + \frac{1}{2\pi}\int_{0}^{\pi}(\pi - x)dx = \frac{3\pi}{4},$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} \pi \cos nx dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\cos nx dx = \frac{1}{\pi n^2}[1 - (-1)^n],$$
$$\text{for } n = 1, 2, \ldots.$$

This last result simplifies to

$$a_{2n-1} = \frac{2}{\pi(2n-1)^2}, \quad a_{2n} = 0, \quad \text{for } n = 1, 2, \ldots.$$

Similarly,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} \pi \sin nx dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\sin nx dx = \frac{(-1)^n}{n}, \quad \text{for } n = 1, 2, \ldots.$$

Substituting the coefficients a_n and b_n into the Fourier series gives

$$f(x) = \frac{3\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} + \sum_{n=1}^{\infty}\frac{(-1)^n \sin nx}{n} \quad \text{for } -\pi \le x \le \pi.$$

To find the harmonics and the amplitude spectrum, it is necessary to group together terms with corresponding frequencies. When this is done $f(x)$ becomes

$$f(x) = \frac{3\pi}{4} + \left(\frac{2}{\pi}\cos x - \sin x\right) + \frac{1}{2}\sin 2x + \left(\frac{2}{9\pi}\cos 3x - \frac{1}{3}\sin 3x\right)$$
$$+ \frac{1}{4}\sin 4x + \left(\frac{2}{25\pi}\cos 5x - \frac{1}{5}\sin 5x\right) + \cdots.$$

This shows, for example, that the fifth harmonic is proportional to

$$\frac{2}{25\pi}\cos 5x - \frac{1}{5}\sin 5x.$$

The amplitudes are

$$A_0 = |a_0| = \frac{3\pi}{4}, \quad A_1 = \left[\left(\frac{2}{\pi}\right)^2 + (-1)^2\right]^{1/2},$$

$$A_2 = \frac{1}{2}, \quad A_3 = \left[\left(\frac{2}{9\pi}\right)^2 + \left(-\frac{1}{3}\right)^2\right]^{1/2},$$

$$A_4 = \frac{1}{4}, \quad A_5 = \left[\left(\frac{2}{25\pi}\right)^2 + \left(-\frac{1}{5}\right)^2\right]^{1/2}, \ldots.$$

In general

$$A_{2n-1} = \frac{1}{(2n-1)}\left[\frac{4}{(2n-1)^2\pi^2} + 1\right]^{1/2} \quad \text{and} \quad A_{2n} = \frac{1}{2n}, \quad \text{for } n = 1, 2, \ldots.$$

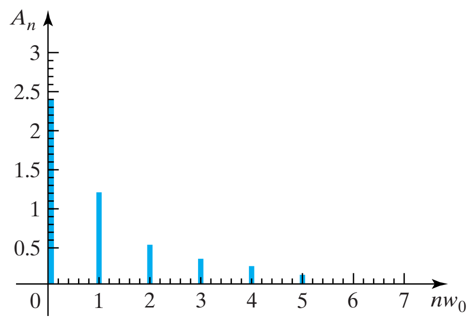


FIGURE 9.13 The amplitude spectrum of $f(x)$ as a function of frequency.

The first few numerical values of the amplitudes are

$A_0 = 2.356,\quad A_1 = 1.185,\quad A_2 = 0.5,\quad A_3 = 0.341,\quad A_4 = 0.25,\quad A_5 = 0.202,$
$A_6 = 0.167, \ldots,$

and the amplitude spectrum of $f(x)$ is shown in Fig. 9.13. In Fig. 9.13 the amplitudes $A_0, A_1, \ldots,$ are represented by vertical lines of length $A_0, A_1, \ldots,$ corresponding to the frequencies $0, 1, 2, \ldots.$

The phases $\delta_n = \text{Arctan}\,(-b_n/a_n)$ are seen to be given by

$$\delta_1 = \text{Arctan}\,(\pi/2),\quad \delta_2 = \text{Arctan}\,(-\infty),\quad \delta_3 = \text{Arctan}\,(3\pi/2),$$
$$\delta_4 = \text{Arctan}\,(-\infty),\quad \delta_5 = \text{Arctan}\,(5\pi/2), \ldots.$$

The negative sign is required in the arctangent functions associated with phases with even suffixes so that when the terms $A_{2n}\cos(2nx + \delta_{2n})$ are expanded, the functions $\sin 2nx$ have a positive sign. ■

Summary

It was shown how a Fourier series can be interpreted in a different way by introducing an angular frequency ω_0, combining sine and cosine terms with similar arguments into a single cosine term with a phase angle, and calling the magnitude of the multiplier of the cosine term the amplitude associated with the cosine term. A discrete plot of amplitude as a function of frequency was then called the amplitude spectrum of the representation. This form of representation is useful in many applications involving vibrations, because when the response of a system is represented in this way, the square of the amplitude is proportional to the energy in the system at that frequency, so the plot shows the distribution of energy as a function of frequency.

EXERCISES 9.5

In the following exercises find the frequency and amplitude spectrum of the given functions.

1. $f(x) = \begin{cases} 0, & -2\pi < x < 0 \\ x, & 0 < x < 2\pi. \end{cases}$

2. $f(x) = x, \quad -\pi/2 < x < \pi/2.$

3. $f(x) = \begin{cases} 1, & -\pi < x < 0 \\ -3, & 0 < x < \pi. \end{cases}$

4. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi. \end{cases}$

5. $f(x) = x^2, \quad -\pi/4 < x < \pi/4.$

9.6 Double Fourier Series

Fourier series representations extend in a natural way to functions $f(x, y)$ of two real variables x and y over the intervals $-L_1 \le x \le L_1$ and $-L_2 \le y \le L_2$, provided f can be represented as a Fourier series in x when y is held constant, and as a Fourier series in y when x is held constant.

To arrive at a double Fourier series representation for $f(x, y)$, we first consider y to be a constant and write $f(x, y)$ as

$$f(x, y) = \sum_{m=0}^{\infty} \left(A_m(y) \cos \frac{m\pi x}{L_1} + B_m(y) \sin \frac{m\pi x}{L_1} \right), \tag{49}$$

and then allow y to vary by replacing the Fourier coefficients $A_m(y)$ and $B_m(y)$ by their Fourier series representations

extending Fourier series to function $f(x, y)$ of two variables

$$A_m(y) = \sum_{n=0}^{\infty} \left(a_{mn} \cos \frac{n\pi y}{L_2} + b_{mn} \sin \frac{n\pi y}{L_2} \right) \tag{50}$$

and

$$B_m(y) = \sum_{n=0}^{\infty} \left(c_{mn} \cos \frac{n\pi y}{L_2} + d_{mn} \sin \frac{n\pi y}{L_2} \right).$$

Substituting (50) into (49) shows $f(x, y)$ can be written as

$$\begin{aligned} f(x, y) = &\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(a_{mn} \cos \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2} + b_{mn} \cos \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2} \right) \\ &+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(c_{mn} \sin \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2} + d_{mn} \sin \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2} \right). \end{aligned} \tag{51}$$

The Fourier coefficients a_{mn} for $m, n = 1, 2, \ldots$ are found by multiplying (51) by $\cos \frac{s\pi x}{L_1}$ and integrating over the interval $-L_1 \le x \le L_1$ to get

$$\begin{aligned} \int_{-L_1}^{L_1} f(x, y) \cos \frac{s\pi x}{L_1} dx = &\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[a_{mn} \cos \frac{n\pi y}{L_2} \int_{-L_1}^{L_1} \cos \frac{m\pi x}{L_1} \cos \frac{s\pi x}{L_1} dx \right] \\ &+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[b_{mn} \sin \frac{n\pi y}{L_2} \int_{-L_1}^{L_1} \cos \frac{m\pi x}{L_1} \cos \frac{s\pi x}{L_1} dx \right] \\ &+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[c_{mn} \cos \frac{n\pi y}{L_2} \int_{-L_1}^{L_1} \sin \frac{m\pi x}{L_1} \cos \frac{s\pi x}{L_1} dx \right] \\ &+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[d_{mn} \sin \frac{n\pi y}{L_2} \int_{-L_1}^{L_1} \sin \frac{m\pi x}{L_1} \cos \frac{s\pi x}{L_1} dx \right]. \end{aligned} \tag{52}$$

The orthogonality of the functions $\cos \frac{m\pi x}{L_1}$ and $\sin \frac{s\pi x}{L_1}$ over the interval $-L_1 \le x \le L_1$ reduces (52) to

$$\int_{-L_1}^{L_1} f(x, y) \cos \frac{s\pi x}{L_1} dx = \sum_{n=0}^{\infty} \left(a_{sn} L_1 \cos \frac{n\pi y}{L_2} + b_{sn} L_1 \sin \frac{n\pi y}{L_2} \right). \tag{53}$$

Multiplication of (53) by $\cos \frac{t\pi y}{L_2}$ followed by integration over the interval $-L_2 \le$

$y \leq L_2$ reduces it further to

$$\int_{-L_2}^{L_2} \left[\int_{-L_1}^{L_1} f(x, y) \cos \frac{s\pi x}{L_1} \right] \cos \frac{t\pi y}{L_2} dy = a_{st} L_1 L_2,$$

so replacing s by m and t by n gives

$$a_{mn} = \frac{1}{L_1 L_2} \int_{-L_2}^{L_2} \int_{-L_1}^{L_1} f(x, y) \cos \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2} dxdy \quad \text{for } m, n = 1, 2, \ldots. \tag{54}$$

The coefficient a_{00} follows by setting $m = n = 0$ in (51) and integrating over the intervals $-L_1 \leq x \leq L_1$ and $-L_2 \leq y \leq L_2$ to give

$$a_{00} = \frac{1}{4L_1 L_2} \int_{-L_2}^{L_2} \int_{-L_1}^{L_1} f(x, y) dxdy. \tag{55}$$

It remains to find the coefficients a_{m0} and a_{0n} for $m, n = 1, 2, \ldots$. Setting $n = 0$ in (53), integrating over $-L_2 \leq y \leq L_2$, and then replacing s by m gives

$$a_{m0} = \frac{1}{2L_1 L_2} \int_{-L_2}^{L_2} \int_{-L_1}^{L_1} f(x, y) \cos \frac{m\pi x}{L_1} dxdy. \tag{56}$$

The coefficients a_{0n} for $n = 1, 2, \ldots$ follow by multiplying (51) by $\cos \frac{t\pi y}{L_1}$, integrating over the interval $-L_2 \leq y \leq L_2$, and then replacing t by n to obtain

$$a_{0n} = \frac{1}{2L_1 L_2} \int_{-L_2}^{L_2} \int_{-L_1}^{L_1} f(x, y) \cos \frac{n\pi y}{L_2} dxdy. \tag{57}$$

Corresponding arguments show that for $m, n = 1, 2, \ldots,$

$$b_{mn} = \frac{1}{L_1 L_2} \int_{-L_2}^{L_2} \int_{-L_1}^{L_1} f(x, y) \cos \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2} dxdy, \tag{58}$$

$$c_{mn} = \frac{1}{L_1 L_2} \int_{-L_2}^{L_2} \int_{-L_1}^{L_1} f(x, y) \sin \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2} dxdy, \tag{59}$$

$$d_{mn} = \frac{1}{L_1 L_2} \int_{-L_2}^{L_2} \int_{-L_1}^{L_1} f(x, y) \sin \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2} dxdy, \tag{60}$$

where

$$b_{m0} = 0, \quad c_{0n} = 0, \quad d_{0n} = 0 \quad \text{and} \quad d_{m0} = 0, \tag{61}$$

because the index zero causes the sine function to vanish in the integrands of the integrals defining these constants.

general and special double Fourier series representations

Thus, the general **double Fourier series representation** of $f(x, y)$ over the interval $-L_1 \leq x \leq L_1$ and $-L_2 \leq y \leq L_2$ is given by

$$\begin{aligned} f(x, y) = &\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(a_{mn} \cos \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2} + b_{mn} \cos \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2} \right) \\ &+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(c_{mn} \sin \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2} + d_{mn} \sin \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2} \right), \end{aligned} \tag{62}$$

where the coefficients a_{mn}, b_{mn}, c_{mn}, and d_{mn} are given by expressions (54) to (61).

The following useful special cases arise according as the function $f(x, y)$ is even or odd in its variables.

Case (a) $f(x, y)$ Is Even in x and y

In this case $f(-x, y) = f(x, y)$ and $f(x, -y) = f(x, y)$, so only the coefficients a_{mn} are nonzero, leading to the **double Fourier cosine series representation**

$$
\begin{aligned}
f(x, y) = a_{00} &+ \sum_{m=1}^{\infty} a_{m0} \cos \frac{m\pi x}{L_1} + \sum_{n=1}^{\infty} a_{0n} \cos \frac{n\pi y}{L_2} \\
&+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \cos \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2}.
\end{aligned}
\tag{63}
$$

As $f(x, y)$ is even in both x and y, both limits of integration in the integrals defining the a_{mn} in (54) to (57) can be changed to give

$$
\begin{aligned}
a_{00} &= \frac{1}{L_1 L_2} \int_0^{L_2} \int_0^{L_1} f(x, y) dx dy \\
a_{m0} &= \frac{2}{L_1 L_2} \int_0^{L_2} \int_0^{L_1} f(x, y) \cos \frac{m\pi x}{L_1} dx dy, \quad m = 1, 2, \dots \\
a_{0n} &= \frac{2}{L_1 L_2} \int_0^{L_2} \int_0^{L_1} f(x, y) \cos \frac{n\pi y}{L_2} dx dy, \quad n = 1, 2, \dots \\
a_{mn} &= \frac{4}{L_1 L_2} \int_0^{L_2} \int_0^{L_1} f(x, y) \cos \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2} dx dy, \quad m, n = 1, 2, \dots .
\end{aligned}
\tag{64}
$$

Case (b) $f(x, y)$ Is Even in x and Odd in y

In this case $f(-x, y) = f(x, y)$ and $f(x, -y) = -f(x, y)$ so only the coefficients b_{mn} are nonzero, leading to the representation

$$
f(x, y) = \sum_{n=1}^{\infty} b_{0n} \sin \frac{n\pi y}{L_2} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn} \cos \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2}.
\tag{65}
$$

As $f(x, y)$ is even only in x, the limits of integration for x in integral (58) defining the coefficients b_{mn} can be changed to give

$$
\begin{aligned}
b_{mn} &= \frac{2}{L_1 L_2} \int_{-L_2}^{L_2} \int_0^{L_1} f(x, y) \cos \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2} dx dy \\
&= \frac{4}{L_1 L_2} \int_0^{L_2} \int_0^{L_1} f(x, y) \cos \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2} dx dy.
\end{aligned}
\tag{66}
$$

Case (c) $f(x, y)$ Is Odd in x and Even in y

In this case $f(-x, y) = -f(x, y)$ and $f(x, -y) = f(x, y)$, so only the coefficients c_{mn} are nonzero, leading to the representation

$$f(x, y) = \sum_{m=1}^{\infty} c_{m0} \sin \frac{m\pi y}{L_1} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn} \sin \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2}. \tag{67}$$

As $f(x, y)$ is even only in y, the limits of integration for y in integral (59) defining the coefficients c_{mn} can be changed to give

$$\begin{aligned} c_{mn} &= \frac{2}{L_1 L_2} \int_{-L_2}^{L_2} \int_{-L_1}^{L_1} f(x, y) \sin \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2} dx dy \\ &= \frac{4}{L_1 L_2} \int_{0}^{L_2} \int_{-L_1}^{L_2} f(x, y) \sin \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2} dx dy. \end{aligned} \tag{68}$$

Case (d) $f(x, y)$ Is Odd in x and y

In this case $f(-x, y) = -f(x, y)$ and $f(x, -y) = -f(x, y)$ so only the coefficients d_{mn} are nonzero, leading to the **double Fourier sine series representation**

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} d_{mn} \sin \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2}. \tag{69}$$

As $f(x, y)$ is odd in both x and y, both limits of integration for x and y in integral (60) defining the coefficients d_{mn} can be changed to give

$$d_{mn} = \frac{4}{L_1 L_2} \int_{0}^{L_2} \int_{0}^{L_1} f(x, y) \sin \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2} dx dy. \tag{70}$$

EXAMPLE 9.15 Find the double Fourier series representation of $f(x, y) = xy$ over $-2 \le x \le 2$ and $-4 \le y \le 4$.

Solution The function $f(x, y)$ is odd in both x and y, so this corresponds to the double Fourier sine series representation of case (d) with $L_1 = 2$ and $L_2 = 4$. From (70) we have

$$\begin{aligned} d_{mn} &= \frac{4}{8} \int_{0}^{4} \int_{0}^{2} xy \sin \frac{m\pi x}{2} \sin \frac{n\pi y}{4} dx dy \\ &= \frac{1}{2} \left[\int_{0}^{2} x \sin \frac{m\pi x}{2} dx \right] \left[\int_{0}^{4} y \sin \frac{n\pi y}{4} dy \right] \\ &= \frac{1}{2} \left[\frac{-4(-1)^m}{m\pi} \right] \left[\frac{-16(-1)^n}{n\pi} \right] = (-1)^{m+n} \frac{32}{mn\pi^2}. \end{aligned}$$

Thus, the required double Fourier sine series representation is

$$f(x, y) = \frac{32}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+n}\frac{1}{mn}\sin\frac{m\pi x}{2}\sin\frac{n\pi y}{4},$$

for $-2 \le x \le 2$ and $-4 \le y \le 4$. Notice that this same expression describes the representation of $f(x, y)$ for $0 \le x \le 2$ and $0 \le y \le 4$. ■

By analogy with the half-range sine and cosine series of Section 9.3, a function $f(x, y)$ defined in a region $0 \le x \le a, 0 \le y \le b$ can be extended to the region $-a \le x \le a, -b \le y \le b$ either as a function that is odd in both x and y, or as one that is even in both x and y. If it is extended as an odd function, case (d) applies and the representation in the first quadrant follows by restricting the result to $0 \le x \le a$, $0 \le y \le b$, whereas if it is extended as an even function, case (a) applies, when the representation is again obtained by restricting the result to $0 \le x \le a, 0 \le y \le b$.

Suppose, for example, a double Fourier sine series representation of $f(x, y) = xy$ is required for $0 \le x \le 2$ and $0 \le y \le 4$. Then extending $f(x, y)$ to the region $-2 \le x \le 2, -4 \le y \le 4$ as a function that is odd in both x and y leads to Example 9.15, so the required representation is given by restricting the double Fourier sine series of Example 9.15 to $0 \le x \le 2$ and $0 \le y \le 4$. Similarly, $f(x, y) = xy$ can be represented by a double Fourier cosine series in $0 \le x \le 2$ and $0 \le y \le 4$ by extending it as $f(x, y) = |x||y|$ for $-2 \le x \le 2$ and $-4 \le y \le 4$. As $f(x, y)$ is even in both x and y, case (a) can be applied and the result again restricted so that $0 \le x \le 2$ and $0 \le y \le 4$.

A typical plot of a double Fourier series approximation to $f(x, y) = xy$ for $0 \le x \le 2$ and $0 \le y \le 4$ provided by a partial sum of the double Fourier sine series in Example 9.15 is shown in Fig. 9.14 for the case with $m = n = 10$. If, instead, the cosine approximation had been used (see Exercise 6), the plot of the corresponding approximation provided by the partial sum with $m = n = 10$ is shown in Fig. 9.15. The convergence of the double cosine series is seen to be the faster of the two.

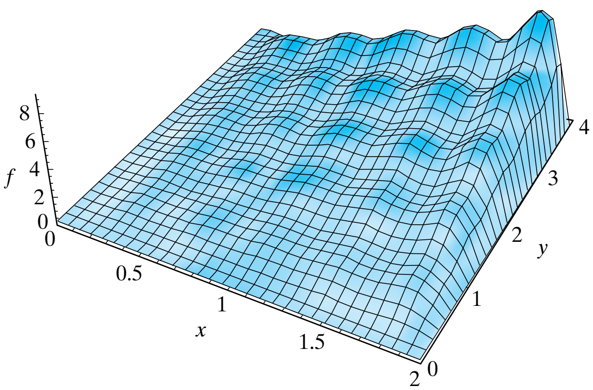


FIGURE 9.14 A double Fourier sine series approximation to $f(x, y) = xy$.

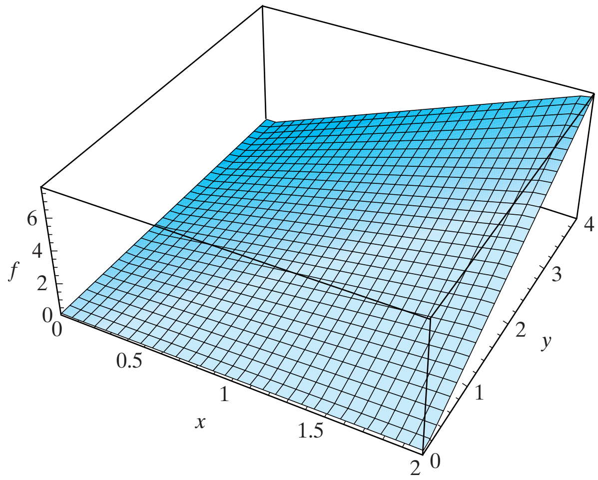


FIGURE 9.15 A double Fourier cosine series approximation to $f(x, y) = xy$.

Summary

It was shown how an ordinary Fourier series representation can be extended in a natural way to the expansion of functions $f(x, y)$ of two variables. After the derivation of the general expansion result, four useful special cases were examined and illustrated by example. Unless $f(x, y)$ is simple, the Fourier series approximation of functions of two variables can require numerical integration when finding the Fourier coefficients, and many terms are usually required to achieve good convergence, so in general it is necessary to perform such calculations and to plot the result by computer.

EXERCISES 9.6

1. By setting $y = 1$ in $f(x, y) = x^2 y$, with $-\pi \le x \le \pi$ and $-\pi \le y \le \pi$, show that the double Fourier series representation of $f(x, y)$ reduces to the ordinary Fourier series representation of $f(x) = x^2$ for $-\pi \le x \le \pi$ given by

$$f(x) = \frac{\pi^2}{3} + 4\sum_{m=1}^{\infty}(-1)^m \frac{\cos mx}{m^2}$$

In Exercises 2 through 9 find and plot double Fourier series partial sum approximations to the given function.

2. $f(x, y) = xy^2$, for $-\pi \le x \le \pi$ and $-\pi \le y \le \pi$.

3. $f(x, y) = x^3 y$, for $-\pi \le x \le \pi$ and $-\pi \le y \le \pi$.

4. $f(x, y) = x^2 y^2$, for $-\pi \le x \le \pi$ and $-\pi \le y \le \pi$.

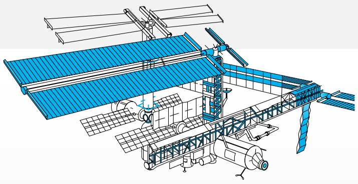
5.* $f(x, y) = \text{sign}(xy)$, for $-\pi \le x \le \pi$ and $-\pi \le y \le \pi$, where sign $u = 1$ if $u > 0$ and sign $u = -1$ if $u < 0$.

6.* $f(x, y) = |xy|$, for $-2 \le x \le 2$ and $-4 \le y \le 4$.

7.* $f(x, y) = \text{sign}\,(xy) + xy$, for $-\pi \le x \le \pi$ and $-\pi \le y \le \pi$.

8.* $f(x, y) = y|\sin x|$, for $-\pi \le x \le \pi$ and $-\pi \le y \le \pi$.

9.* Extend $f(x, y) = xy^2$, for $0 \le x \le \pi$ and $0 \le y \le \pi$, to $-\pi \le x \le \pi$ and $-\pi \le y \le \pi$ as an odd function, and hence find a double Fourier sine series representation of $f(x, y)$ for $0 \le x \le \pi$ and $0 \le y \le \pi$.



CHAPTER 9
TECHNOLOGY PROJECTS

The purpose of these projects is to use computer algebra to generate Fourier series for continuous and discontinuous functions, to use computer graphics to examine their convergence to the functions they represent, and to explore the nature of the Gibbs phenomenon.

Project 1

Finding Fourier Series and Plotting Partial Sums

Use computer algebra to find the first 11 terms $a_0, a_1, \ldots, a_5, b_1, b_2, \ldots, b_5$ of the Fourier series of

$$f(x) = (\pi^2 - x^2)e^{-x} \sin x \quad \text{for } -\pi \le x \le \pi.$$

Plot the approximation to $f(x)$ obtained by using (a) the terms involving a_0, a_1, a_2, b_1, and b_2 and (b) the 11 terms involving $a_0, \ldots, a_5, b_1, \ldots, b_5$ in the partial sum approximation, and compare the results with the graph of $f(x)$.

Project 2

Examining the Gibbs Phenomenon

Use computer algebra to find the Fourier series representation of the function

$$f(x) = \begin{cases} \sin x - 1, & -\pi < x < 0 \\ \sin x + 1, & 0 < x < \pi. \end{cases}$$

By plotting the partial sum representations of $f(x)$ using different numbers of terms, demonstrate the persistence of the overshoot and undershoot caused by the Gibbs phenomenon as the number of terms in the approximation increases.

Project 3

The Complex Fourier Series

Use computer algebra with the complex Fourier series representation of a function to verify the coefficients c_n and c_{2n-1} found in Example 9.12. Plot different partial sum approximations to $f(x)$ and, as in Project 2, demonstrate the persistence of the Gibbs phenomena as the number of terms in the partial sum approximation increases.

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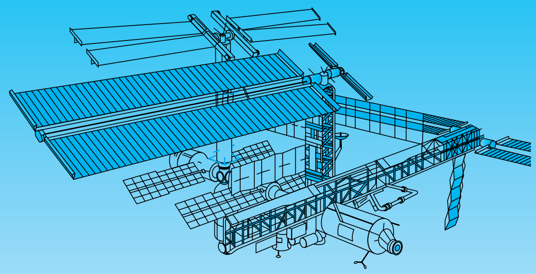
CHAPTER 10



Fourier Integrals and the Fourier Transform

Fourier series enable functions and solutions of linear systems defined over a finite interval to be represented as an infinite series of sines and cosines. This suffices for many physical problems, but often the interval involved is either semi-infinite or infinite, in which case a somewhat different representation becomes necessary. This happens, for example, when working with the partial differential equations that describe heat conduction and diffusion in a half-space for which Fourier series cannot be used.

The Fourier integral can be regarded as the limiting case of a Fourier series representation of a function $f(x)$ defined over an interval $-L < x < L$ as $L \to \infty$. The meaning of the integral representation when the function to be represented is discontinuous is considered, and the special cases of the sine and cosine integral representations are introduced.

Fourier sine and cosine transforms are considered, tables of their transform pairs are given, and the transform of derivatives is discussed. In anticipation of Chapter 18, an application of the Fourier transform is made to the problem of the one-dimensional time dependent heat equation.

10.1 The Fourier Integral

A Fourier series has been shown to represent an arbitrary function $f(x)$ over an interval $-L \le x \le L$, and because the series is periodic with period $2L$ the representation of $f(x)$ in this fundamental interval is repeated by periodicity for all x outside the interval. However, even if $f(x)$ is defined outside the fundamental interval, it does not necessarily follow that the function and its periodic extensions coincide outside the interval. This means that if a nonperiodic function is to be represented over an arbitrarily large interval, some generalization of a Fourier series is required.

Letting $L \to \infty$ in a Fourier series leads to the introduction of a different type of representation called a **Fourier integral representation**, where the function $f(x)$ is defined for all x and need not be periodic. This representation forms the basis of an integral transform called the **Fourier transform** that is similar to the Laplace transform. As with the Laplace transform, one of the the main uses of the Fourier transform is in the solution of differential equations.

The derivation of the Fourier integral representation given here is heuristic, because a rigorous one requires techniques that are not needed elsewhere in the book. We start from the definition of a Fourier series of $f(x)$ over an interval $-L \leq x \leq L$ given in (18) and (19) of Section 9.1 by writing

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{1}$$

where

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx \quad \text{for } n = 1, 2, \ldots. \end{aligned} \tag{2}$$

Substituting the Fourier coefficients (2) into Fourier series (1) allows it to be written in the integral form

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(u)du + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(u)\cos\frac{n\pi(u-x)}{L}du. \tag{3}$$

To proceed further, if the representation is to remain valid as $L \to \infty$ the first term must not become either infinite or indeterminate. This will certainly be true if $\lim_{L\to\infty}\int_{-L}^{L}|f(x)|dx$ is finite, because then the integral involved in the first term will be *absolutely convergent* and the first term in (3) will vanish in the limit as $L \to \infty$. From now on we will assume this condition to be satisfied. We can now write (3) as

$$f(x) = \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(u)\cos\frac{n\pi(u-x)}{L}du. \tag{4}$$

It is from this point onward that our derivation of the Fourier integral representation becomes heuristic, because the arguments used to convert (4) to an integral over the interval $(-\infty, \infty)$ are merely intuitive. A careful examination of the convergence of the double integral involved would be necessary to provide a rigorous justification.

Setting $\Delta_n\omega = \pi/L$, and defining the frequency $\omega_n = n\pi/L$, allows (4) to be rewritten as

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\Delta_n\omega\int_{-L}^{L} f(u)\cos[\omega_n(u-x)]du. \tag{5}$$

Examination of (5) suggests it is equivalent to the pre-limit sum approximation used in the definition of the definite (Riemann) integral of the function

$$F(u) = \frac{1}{\pi}\int_{-L}^{L} f(u)\cos\omega(u-x)du.$$

Using this last result in (5), and proceeding to the limit as $L \to \infty$, we obtain

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty} d\omega\int_{-\infty}^{\infty} f(u)\cos\omega(u-x)du, \tag{6}$$

the Fourier integral representation

which is called the **Fourier integral representation** of $f(x)$.

By defining the functions $A(\omega)$ and $B(\omega)$ as

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\omega u du \quad \text{and} \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\omega u du, \tag{7}$$

the Fourier integral representation in (6) can be written in the simpler form

$$f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega. \tag{8}$$

The convergence properties of Fourier series recorded in Theorem 9.1 can be shown to be transferred to the Fourier integral representation of $f(x)$ if, in addition to the integral of $f(x)$ being absolutely convergent over $(-\infty, \infty)$, it also satisfies certain other conditions. These conditions, called **Dirichlet conditions**, are as follows:

Dirichlet conditions

(i) In any finite interval $f(x)$ has only a finite number of maxima and minima
(ii) In any finite interval $f(x)$ has only a finite number of bounded jump discontinuities and no infinite jump discontinuities.

We now state the following theorem for the Fourier integral without proof.

PETER GUSTAV LEJEUNE DIRICHLET (1805–1859)
A German mathematician who studied under Gauss, was the son-in-law of Jacobi and succeeded Gauss as Professor of Mathematics at Göttingen. He did much to make some of the more abstruse contributions by Gauss better understood. His most important contributions to mathematics were his major contribution to the understanding of the convergence of Fourier series, and his work on number theory and the theory of potential.

THEOREM 10.1

the fundamental Fourier integral theorem

Fourier integral theorem Let $f(x)$ satisfy Dirichlet conditions, and suppose the (sufficiency) conditions that $f(x)$ be both integrable and absolutely integrable over the interval $-\infty < x < \infty$ are both satisfied, so each of the integrals $\int_{-\infty}^{\infty} f(x)dx$ and $\int_{-\infty}^{\infty} |f(x)|dx$ exists. Then

$$\frac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{\pi}\int_0^{\infty} d\omega \int_{-\infty}^{\infty} f(u)\cos\omega(u-x)du$$

or, equivalently,

$$\frac{1}{2}[f(x+0) + f(x-0)] = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega,$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\omega u du \quad \text{and} \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\omega u du.$$ ■

EXAMPLE 10.1

Find the Fourier integral representation of $f(x) = e^{-|x|}$.

Solution The function $e^{-|x|}$ satisfies the Dirichlet conditions, and $\int_{-\infty}^{\infty} |e^{-|x|}|dx = 2$, so the integral of $f(x) = e^{-|x|}$ over $(-\infty, \infty)$ is absolutely convergent. This confirms that $f(x) = e^{-|x|}$ has a Fourier integral representation.

The function $e^{-|x|}$ is even in x, so $e^{-|u|}\cos\omega u$ is also even, and

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-|u|}\ \cos\omega u du = \frac{2}{\pi}\int_{0}^{\infty} e^{-u}\ \cos\omega u du = \frac{2}{\pi(1+\omega^2)}.$$

As the function $e^{-|u|}\sin\omega u$ is odd in u,

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-|u|}\sin\omega u du = \frac{2}{\pi}\int_{0}^{\infty} e^{-u}\sin\omega u du = 0,$$

so from (8) the Fourier integral representation of $e^{-|x|}$ is seen to be

$$e^{-|x|} = \frac{2}{\pi}\int_{0}^{\infty}\frac{\cos\omega x}{1+\omega^2}d\omega.$$

■

EXAMPLE 10.2

Find the Fourier integral representation of

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x < 0 \end{cases}$$

and use Theorem 10.1 to find the value of the resulting integral when (a) $x < 0$, (b) $x = 0$, and (c) $x > 0$.

Solution The function $f(x)$ satisfies the Dirichlet conditions and the integral $\int_{-\infty}^{\infty}|f(x)|dx = \int_{0}^{\infty}e^{-x}dx = 1$, so as the conditions of Theorem 10.1 are satisfied the function has a Fourier integral representation.

We have

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\omega u du = \frac{1}{\pi}\int_{0}^{\infty} e^{-u}\cos\omega u du = \frac{1}{\pi(1+\omega^2)}$$

and

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\omega u du = \frac{1}{\pi}\int_{0}^{\infty} e^{-u}\sin\omega u du = \frac{\omega}{\pi(1+\omega^2)}.$$

Substituting into (8) shows the Fourier integral representation to be

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\frac{\cos\omega x + \omega\ \sin\ \omega x}{1+\omega^2}d\omega \quad \text{for } -\infty < x < \infty.$$

Applying the results of Theorem 10.1 to this integral, we find that

$$\pi f(x) = \int_{0}^{\infty}\frac{\cos\omega x + \omega\ \sin\ \omega x}{1+\omega^2}d\omega = \begin{cases} 0, & x < 0 \\ \pi/2, & x = 0 \\ \pi e^{-x}, & x > 0. \end{cases}$$

When $x = 0$, this last result is seen to reduce to the familiar definite integral

$$\int_{0}^{\infty}\frac{d\omega}{1+\omega^2} = \frac{\pi}{2}.$$

■

Special forms of the Fourier integral representation arise according to whether $f(x)$ is even or odd. When $f(x)$ is an even function, $f(u)\sin\omega u$ is an odd function of u,

so $B(\omega) \equiv 0$ and

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(u)\cos\omega u du, \tag{9}$$

so that (8) simplifies to the **Fourier cosine integral representation** of $f(x)$

$$f(x) = \int_0^\infty A(\omega)\cos\omega x d\omega. \tag{10}$$

Similarly, when $f(x)$ is an odd function, $f(u)\cos\omega u$ is an odd function of u, so $A(\omega) \equiv 0$ and

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(u)\sin\omega u du, \tag{11}$$

causing (8) to simplify to the **Fourier sine integral representation** of $f(x)$ given by

$$f(x) = \int_0^\infty B(\omega)\sin\omega x d\omega. \tag{12}$$

different Fourier integral representations

Summary of Fourier integral representations

(a) An *arbitrary* function $f(x)$ satisfying the conditions of Theorem 10.1 has the **general Fourier integral representation**

$$\frac{1}{2}[f(x+0) + f(x-0)] = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega. \tag{13}$$

(b) An *even* function $f(x)$ satisfying the conditions of Theorem 10.1 has the **Fourier cosine integral representation**

$$\frac{1}{2}[f(x+0) + f(x-0)] = \int_0^\infty A(\omega)\cos\omega x d\omega. \tag{14}$$

(c) An *odd* function $f(x)$ satisfying the conditions of Theorem 10.1 has the **Fourier sine integral representation**

$$\frac{1}{2}[f(x+0) + f(x-0)] = \int_0^\infty B(\omega)\sin\omega x d\omega, \tag{15}$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(u)\cos\omega u du \quad \text{and} \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(u)\sin\omega u du. \tag{16}$$

Summary

The Fourier integral representation of a function $f(x)$ was introduced as the natural extension of a Fourier series representation as the interval of the representation extends to become the interval $-\infty < x < \infty$. A fundamental representation theorem was given and illustrated by example, and some useful special cases of the theorem were considered.

EXERCISES 10.1

Find the Fourier integral representation of the given functions.

1. The rectangular pulse function $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ (Fig. 10.1).

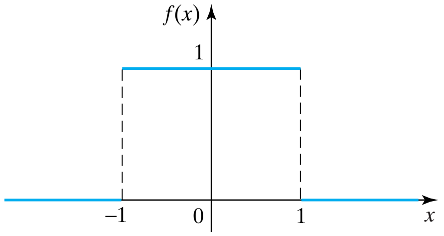


FIGURE 10.1 The rectangular pulse function.

2. The triangular function

$$f(x) = \begin{cases} 0, & |x| > a \\ b\left(1 + \dfrac{x}{a}\right), & -a \le x \le 0 \\ b\left(1 - \dfrac{x}{a}\right), & 0 \le x \le a \end{cases} \quad \text{(Fig. 10.2).}$$

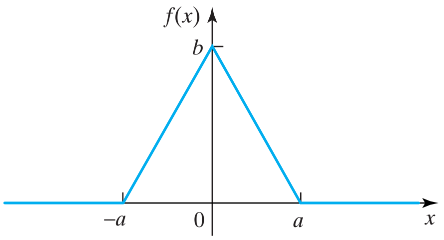


FIGURE 10.2 The triangular function.

3. $f(x) = \begin{cases} 0, & |x| > a \\ bx/a, & -a \le x \le a \end{cases}$ (Fig. 10.3).

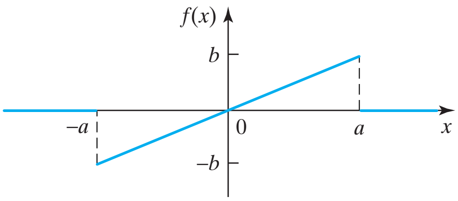


FIGURE 10.3 The truncated straight line function.

4. $f(x) = \begin{cases} 0, & x \le 0 \\ \sin x, & 0 \le x \le \pi \\ 0, & x \ge \pi \end{cases}$ (Fig. 10.4).

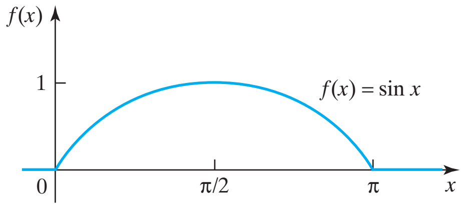


FIGURE 10.4 The asymmetric truncated sine function.

5. $f(x) = \begin{cases} (\pi/2)\cos x, & |x| < \pi/2 \\ 0, & |x| > \pi/2 \end{cases}$ (Fig. 10.5).

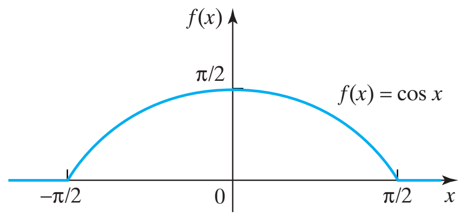


FIGURE 10.5 The truncated cosine function.

6. $f(x) = \begin{cases} (\pi/2)\sin x, & |x| < \pi/2 \\ 0, & |x| > \pi/2 \end{cases}$ (Fig. 10.6).

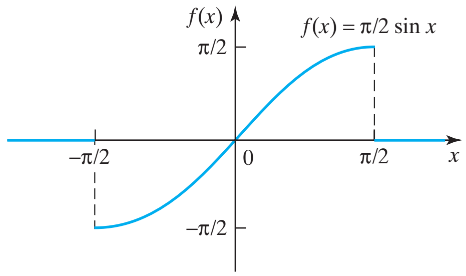


FIGURE 10.6 The truncated sine function.

7. $f(x) = \begin{cases} 0, & x < 0 \\ \cos x, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$ (Fig. 10.7).

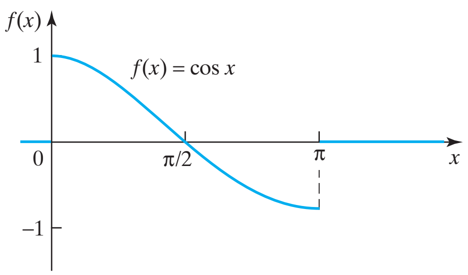


FIGURE 10.7 The asymmetric truncated cosine function.

8. The hump function $f(x) = 1/(1 + x^2)$ (Fig. 10.8). (Hint: Use the result of Example 10.16 with a change of notation.)

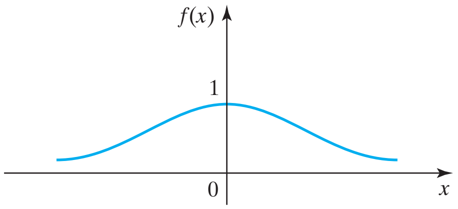


FIGURE 10.8 The hump function.

10.2 The Fourier Transform

The starting point for the development of the *Fourier transform* is the complex form of the Fourier integral representation of a function $f(x)$. To derive this representation in which $f(x)$ is defined over the interval $(-\infty, \infty)$, we substitute into (8) of Section 10.1 the expressions for $A(\omega)$ and $B(\omega)$ given in (7) to obtain

$$\begin{aligned}\frac{1}{2}[f(x+0) + f(x-0)] &= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(u)[\cos\omega u \cos\omega x + \sin\omega u \sin\omega x]du\right]d\omega \\ &= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(u)\cos\{\omega(u-x)\}du\right]d\omega \\ &= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(u)\cos\{\omega(x-u)\}du\right]d\omega,\end{aligned}$$

where we have used the result $\cos\omega(u - x) = \cos\omega(x - u)$.

As the integrand in the last integral is an even function of ω, the interval of integration with respect to ω can be doubled and the result compensated by the introduction of a multiplicative factor 1/2 to give

$$\frac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^{\infty} f(u)\cos\omega(x-u)du\right]d\omega. \tag{17}$$

The function $\sin\omega(x - u)$ is an odd function of ω, so it follows directly that

$$0 = \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^{\infty} f(u)\sin\{\omega(x-u)\}du\right]d\omega. \tag{18}$$

the complex Fourier integral representation

Multiplying equation (18) by i, adding the result to equation (17), and using the Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$, we arrive at the **complex Fourier integral**

representation

$$\frac{1}{2}[f(x+0)+f(x-0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\exp\{i\omega(x-u)\}du\right]d\omega. \tag{19}$$

The brackets in (17) to (19) were retained to clarify the order in which the integrations are performed, but they are usually omitted in (19), which then becomes

$$\frac{1}{2}[f(x+0)+f(x-0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\exp\{i\omega(x-u)\}du\,d\omega. \tag{20}$$

Clearly, the left-hand side of (20) reduces to $f(x)$ wherever the function is continuous.

To arrive at the definitions of a Fourier transform and its inverse we write the factor $\exp\{i\omega(x-u)\}$ in (19) (equivalently (20)) as the product $\exp\{i\omega x\}\cdot\exp\{-i\omega u\}$. Then, as the inner integral only involves integration with respect to u, we rewrite (19) as

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\{i\omega x\}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\exp\{-i\omega u\}du\right]d\omega, \tag{21}$$

where the left-hand side is to be replaced by $(1/2)[f(x+0)+f(x-0)]$ whenever $f(x)$ is discontinuous.

If we now define the function $F(\omega)$ as

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\exp\{-i\omega u\}du,$$

then because u is a dummy variable it can be replaced by x and the result rewritten as

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\exp\{-i\omega x\}dx, \tag{22}$$

so that (19) becomes

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)\exp\{i\omega x\}d\omega. \tag{23}$$

Fourier transforms and transform pairs

The function $F(\omega)$ in (22) is called the **Fourier transform** of $f(x)$, or sometimes the **exponential Fourier transform**, and because integral (23) recovers $f(x)$ from $F(\omega)$ it is called the **inversion integral** for the Fourier transform. As with the Laplace transform, when working with the Fourier transform the function $f(x)$ and the associated Fourier transform $F(\omega)$ are called a **Fourier transform pair**. A short table of Fourier transform pairs is to be found at the end of this section.

Various other notations are used to indicate the Fourier transform of $f(x)$, the most common of which involves representing it by $\hat{f}(\omega)$, so in terms of the notation used here, $\hat{f}(\omega) = F(\omega)$.

Another notation that is often useful involves representing the Fourier transform of $f(x)$ by $\mathcal{F}\{f(x)\}$, so that $\mathcal{F}\{f(x)\} = F(\omega)$, and when this notation is used the inverse Fourier transform is written $\mathcal{F}^{-1}\{F(\omega)\} = f(x)$. In what follows a function to be transformed is denoted by a lowercase letter, and the corresponding uppercase letter is then used to denote its Fourier transform. So, for example, $\mathcal{F}\{g(x)\} = G(\omega)$ and $\mathcal{F}\{h(x)\} = H(\omega)$.

The choice of the normalizing factors $1/\sqrt{2\pi}$ in integrals (22) and (23) is optional, and it is chosen here to introduce as much symmetry as possible into the definitions of a Fourier transform and its inverse. All that is required of the normalizing factors is that their product be $1/(2\pi)$, so in many reference works the factor $1/\sqrt{2\pi}$ in (22) is replaced by 1, while the factor $1/\sqrt{2\pi}$ in (23) is replaced by $1/(2\pi)$. It is impossible to achieve complete symmetry in the definitions of a Fourier integral and its inverse because the exponential factor occurs with opposite signs in (22) and (23).

When Fourier transforms listed in reference works are used, another source of confusion can arise because sometimes the signs in the exponential factors occurring in integrals (22) and (23) are interchanged. When this happens a Fourier transform obtained using this sign convention can be converted to the one used here by reversing the sign of ω. However, each definition of the Fourier transform and the corresponding inversion integral conform to the general pattern

$$\begin{aligned} \mathcal{F}\{f(x)\} &= \frac{k}{2\pi}\int_{-\infty}^{\infty} f(x)\exp\{\pm i\omega x\}dx \quad \text{and} \\ \mathcal{F}^{-1}\{F(\omega)\} &= \frac{1}{k}\int_{-\infty}^{\infty} F(\omega)\exp\{\mp i\omega x\}d\omega, \end{aligned} \tag{24}$$

where k is an arbitrary scale factor.

In view of the different conventions that are in use, when working with Fourier transforms and referring to reference works, it is essential that the normalizing factor k and the sign convention employed in the exponential factors be established before any use is made of the results.

When we considered the convergence of Fourier series, the Riemann–Lebesgue lemma was established the results of which were that

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos nx dx = \lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0. \tag{25}$$

A limiting argument similar to the one used in Section 10.1 when deriving the Fourier integral representation of $f(x)$ shows that, provided $f(x)$ has a Fourier transform,

$$\lim_{|\omega|\to\infty}\int_{-\infty}^{\infty} f(x)\cos\omega x dx = \lim_{|\omega|\to\infty}\int_{-\infty}^{\infty} f(x)\sin\omega x dx = 0. \tag{26}$$

As the Fourier transform $F(\omega)$ of $f(x)$ can be written

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} f(x)\cos\omega x dx - i\int_{-\infty}^{\infty} f(x)\sin\omega x dx\right], \tag{27}$$

an application of limits (26) in (27) establishes the important property of a Fourier transform that

$$\lim_{|\omega|\to\infty} F(\omega) = 0. \tag{28}$$

EXAMPLE 10.3 Find the Fourier transforms of

(a) $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a, \end{cases}$ (b) $g(x) = \begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise}, \end{cases}$ (c) $p(x) = \dfrac{1}{x^2 + a^2}$

by making use of the standard integral $\int_{-\infty}^{\infty} \frac{\cos \omega x}{x^2+a^2} dx = \frac{\pi}{a} e^{-|\omega|a}$ $(a > 0)$ and (d) $q(x) = \begin{cases} e^{iax}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$. In each case confirm that the Fourier transform vanishes as $\omega \to \pm\infty$.

Solution

(a)

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-i\omega x} dx = \frac{1}{\omega\sqrt{2\pi}} \left[\frac{e^{i\omega a} - e^{-i\omega a}}{i}\right]$$

$$= \frac{1}{\omega}\sqrt{\frac{2}{\pi}} \left[\frac{e^{i\omega a} - e^{-i\omega a}}{2i}\right] = \sqrt{\frac{2}{\pi}} \frac{\sin \omega a}{\omega}.$$

As $\sin \omega a$ is bounded, it follows directly that $\lim_{|\omega|\to\infty} F(\omega) = 0$.

(b)

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{0}^{a} e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \left(\frac{1 - e^{-i\omega a}}{i\omega}\right).$$

As the numerator of $G(\omega)$ is bounded, it follows that $\lim_{|\omega|\to\infty} G(\omega) = 0$. This example shows that although $f(x)$ may be real, its Fourier transform can be complex.

(c)

$$P(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{x^2 + a^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\cos \omega x}{x^2 + a^2} dx - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sin \omega x}{x^2 + a^2} dx.$$

The integrand of the second integral is odd, so the value of the integral is zero. Using the standard result

$$\int_{-\infty}^{\infty} \frac{\cos \omega x}{x^2 + a^2} dx = \frac{\pi}{a} e^{-|\omega|a}$$

in the remaining integral on the right, we find that

$$P(\omega) = \sqrt{\frac{\pi}{2}} \frac{e^{-|\omega|a}}{a} \quad (a > 0).$$

In this case the factor $e^{-|\omega|a}$ ensures that $\lim_{|\omega|\to\infty} P(\omega) = 0$.

(d)

$$Q(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{1} e^{-i(\omega - a)x} dx$$

$$= \frac{i}{\sqrt{2\pi}} \left(\frac{1 - e^{-i(\omega - a)}}{a - \omega}\right).$$

As the numerator of the Fourier transform is bounded, the denominator causes the transform to vanish as $|\omega| \to \infty$. This example shows that a complex function can also have a Fourier transform and, in general, that the transform will be complex. ■

the main operational properties of Fourier transforms

The fundamental properties contained in Theorems 10.2 to 10.8 that follow are called **operational properties** of the Fourier transform. Familiarity with these properties is essential, because they simplify calculations involving Fourier transforms and can lead to results that are difficult to obtain without their use.

THEOREM 10.2

Linearity of the Fourier transform Let the functions $f(x)$ and $g(x)$ have the respective Fourier transforms $F(\omega)$ and $G(\omega)$, and let a and b be arbitrary constants. Then

$$\mathcal{F}\{af(x) + bg(x)\} = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}.$$

Proof As the Fourier integral involves the operation of integration, the linearity property of the transform follows directly from the linearity property of the definite integral. ■

Theorem 10.2 is important when the Fourier transform of a sum of functions is required, because it is this result that allows each term involved in the sum to be transformed separately before the results are added.

EXAMPLE 10.4

Find the Fourier transform of $3f(x) - 2g(x)$, where $f(x)$ and $g(x)$ are the functions in (a) and (b) of Example 10.3.

Solution Using the results of Example 10.3 and applying Theorem 10.2, we have

$$\begin{aligned}\mathcal{F}\{3f(x) - 2g(x)\} &= 3\mathcal{F}\{f(x)\} - 2\mathcal{F}\{g(x)\}\\ &= \sqrt{\frac{2}{\pi}}\left\{\frac{3\sin\omega a}{\omega} - \left(\frac{1 - e^{-i\omega a}}{i\omega}\right)\right\}.\end{aligned}$$ ■

THEOREM 10.3

Fourier transform of a derivative of $f(x)$ Let $f(x)$ be a continuous function of x with the property that $\lim_{|x|\to\infty} f(x) = 0$, and such that $f'(x)$ is absolutely integrable over $(-\infty, \infty)$. Then:

(a) $\mathcal{F}\{f'(x)\} = i\omega F(\omega)$.

(b) For all n such that the derivatives $f^{(r)}(x)$ with $r = 1, 2, \ldots, n$ satisfy Dirichlet conditions, are absolutely integrable over $(-\infty, \infty)$, and $\lim_{|x|\to\infty} f^{(n-1)}(x) = 0$,

$$\mathcal{F}\{f^{(n)}(x)\} = (i\omega)^n F(\omega),$$

where $f^{(n)}(x) = d^n f/dx^n$.

Proof

(a) Integration by parts coupled with the condition that $\lim_{|x|\to\infty} f(x) = 0$ gives

$$\begin{aligned}\mathcal{F}\{f'(x)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{-i\omega x}dx\\ &= \frac{1}{\sqrt{2\pi}}\left[f(x)e^{-i\omega x}\Big|_{-\infty}^{\infty} - (-i\omega)\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx\right]\\ &= i\omega\ \mathcal{F}\{f(x)\} = i\omega F(\omega),\end{aligned}$$

where the term $f(x)e^{-i\omega x}|_{-\infty}^{\infty}$ vanishes because of the condition $\lim_{|x|\to\infty} f(x) = 0$.

(b) The second part of the theorem follows by repeated application of result (a), and the conditions imposed on $f^{(n)}(x)$ are necessary to ensure that its Fourier transform exists. ■

EXAMPLE 10.5 Find the Fourier transform of $p'(x)$ from the Fourier transform of $p(x)$, where $p(x)$ is the function in Example 10.3(c).

Solution It was shown in Example 10.3(c) that $P(\omega) = \sqrt{\frac{\pi}{2}}\frac{e^{-|\omega|a}}{a}$, so it follows from Theorem 10.3 (a) that $\mathcal{F}\{p'(x)\} = i\omega P(\omega) = i\omega\sqrt{\frac{\pi}{2}}\frac{e^{-|\omega|a}}{a}$. ■

THEOREM 10.4 **Fourier transform of $x^n f(x)$** Let $f(x)$ be a continuous and differentiable function with an n times differentiable Fourier transform $F(\omega)$. Then

(a) $$\mathcal{F}\{xf(x)\} = i\frac{d}{d\omega}[F(\omega)]$$

and

(b) $$\mathcal{F}\{x^n f(x)\} = i^n \frac{d^n}{d\omega^n}[F(\omega)],$$

for all n such that $\lim_{|\omega|\to\infty} F^{(n)}(\omega) = 0$.

Proof The proof of the theorem follows directly by the application of *Leibniz's rule* that governs differentiation under the integral sign. The rule may be stated as follows:

Leibniz' rule: Let $f(x, \omega)$ and $\partial f/\partial\omega$ be continuous functions of their variables with $-\infty < x < \infty$ and $-\infty < \omega < \infty$. Furthermore, let $\int_{-\infty}^{\infty} |f(x, \omega)|dx$ be finite and $|\partial f/\partial\omega| \le h(x)$ where $h(x)$ is piecewise continuous and such that $\int_{-\infty}^{\infty} h(x)dx$ is finite. Then

$$\frac{d}{d\omega}\int_{-\infty}^{\infty} f(x, \omega)dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial\omega}[f(x, \omega)]dx.$$

(a) Using Leibniz' rule to differentiate the Fourier transform of $f(x)$, we obtain

$$\frac{d}{d\omega}[F(\omega)] = \frac{1}{\sqrt{2\pi}}\frac{d}{d\omega}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xf(x)e^{-i\omega x}dx.$$

The required result follows from this after multiplication by i, because the expression on the right is then $\mathcal{F}\{xf(x)\}$.

(b) The proof for the case when $n > 1$ follows by repeated application of result (a). The conditions imposed on $x^n f(x)$ and $F(\omega)$ are necessary to ensure the existence of the Fourier transform. ■

THEOREM 10.5 **Fourier transform of $x^m f^{(n)}(x)$** Let $f(x)$ be a continuous n times differentiable function. Furthermore, let $x^m f^{(r)}(x)$ for $r = 1, 2, \dots, n$ satisfy Dirichlet conditions and be absolutely integrable over $(-\infty, \infty)$, and let $\omega^n F(\omega)$ possess an m times differentiable inverse Fourier transform. Then, provided $\lim_{|x|\to\infty} f^{(n-1)}(x) = 0$,

$$\mathcal{F}\left\{x^m f^{(n)}(x)\right\} = (i)^{m+n}\frac{d^m}{d\omega^m}[\omega^n F(\omega)].$$

Proof The result follows directly by combining Theorems 10.3 and 10.4, because

$$\mathcal{F}\{x^m f^{(n)}(x)\} = (i)^m \frac{d^m}{d\omega^m}\mathcal{F}\{f^{(n)}(x)\} = (i)^{m+n}\frac{d^m}{d\omega^m}[\omega^n F(\omega)].$$

The conditions imposed on $x^m f^{(n)}(x)$ and $\omega^n F(\omega)$ are necessary to ensure the existence of the Fourier transform. ■

The examples that follow illustrate how Theorems 10.3 to 10.5 may be used to find the Fourier transforms of more complicated functions.

EXAMPLE 10.6 Find the Fourier transform of $f(x) = \exp(-a^2x^2)(a > 0)$.

Solution The function $f(x)$ is continuous and differentiable for all x and

$$\int_{-\infty}^{\infty} |\exp(-a^2x^2)|dx = \int_{-\infty}^{\infty} \exp(-a^2x^2)dx = \frac{1}{a}\int_{-\infty}^{\infty} \exp(-u^2)du = \frac{\sqrt{\pi}}{a},$$

where we have made use of the standard integral $\int_{-\infty}^{\infty} \exp(-u^2)du = \sqrt{\pi}$. This shows that $f(x)$ is absolutely integrable over the interval $(-\infty, \infty)$, and so $f(x)$ has a Fourier transform. A straightforward calculation establishes that $f(x)$ satisfies the differential equation

$$f' + 2a^2xf = 0.$$

Taking the Fourier transform of this equation using Theorem 10.2 gives

$$\mathcal{F}\{f'(x)\} + 2a^2\mathcal{F}\{xf(x)\} = 0.$$

Applying Theorem 10.3 to the first term and Theorem 10.4 to the second term and cancelling a factor i reduces this to the variables separable equation for $F(\omega)$,

$$2a^2F' + \omega F = 0, \quad \text{where } F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp(-a^2x^2)e^{-i\omega x}dx.$$

When variables are separated, the equation becomes

$$\int \frac{F'}{F}d\omega = -\frac{1}{2a^2}\int \omega d\omega,$$

so

$$\ln F(\omega) = -\frac{\omega^2}{4a^2} + \ln A, \quad \text{or} \quad F(\omega) = A\exp\left[-\frac{\omega^2}{4a^2}\right],$$

where, for convenience, the arbitrary integration constant has been written in the form $\ln A$. To determine A we use the fact that $A = F(0)$, but

$$F(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp(-a^2x^2)dx = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{\pi}}{a} = \frac{1}{a\sqrt{2}},$$

and so

$$\mathcal{F}\{\exp(-a^2x^2)\} = F(\omega) = \frac{1}{a\sqrt{2}}\exp\left\{-\frac{\omega^2}{4a^2}\right\} \quad (a > 0).$$ ■

EXAMPLE 10.7

finding the Fourier transform of a function defined by a differential equation

Find the Fourier transform of the Bessel function $J_0(x)$.

Solution The Bessel function $J_0(x)$ does not satisfy the absolute integrability condition found in Theorem 10.1. However, this is merely a sufficient condition that ensures the existence of the Fourier transform of a function $f(x)$, though not a necessary one. Functions exist that possess a Fourier transform even though this condition is violated, and $J_0(x)$ is such a function. The function $f(x) = J_0(x)$ is an even function that is defined for all x and satisfies Bessel's differential equation of order zero

$$xf'' + f' + xf = 0.$$

Taking the Fourier transform of the differential equation by using Theorem 10.2 and then applying Theorem 10.5 to the first term, Theorem 10.3 to the second term, and Theorem 10.4 to the last term, we find, after the cancellation of a factor i and the combination of terms, that

$$(1-\omega^2)F' - \omega F = 0, \quad \text{where } F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} J_0(x)e^{-i\omega x}dx.$$

This is a linear first order variables separable differential equation that can be written

$$\int \frac{F'}{F}d\omega = \int \frac{\omega}{1-\omega^2}d\omega,$$

so integration gives

$$\ln F(\omega) = -\frac{1}{2}\ln(1-\omega^2) + \ln A, \quad \text{or} \quad F(\omega) = \frac{A}{(1-\omega^2)^{1/2}}, \quad \text{with } 0 < \omega^2 < 1.$$

In this equation, the arbitrary integration constant has again been written in the form $\ln A$, and the restriction on ω^2 is necessary because the real logarithmic function is not defined for negative arguments.

To determine A we use the fact that $A = F(0)$, together with the standard result $\int_0^\infty J_0(x)dx = 1$ and the fact that $J_0(x)$ is an even function, to obtain

$$A = F(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} J_0(x)dx = \frac{2}{\sqrt{2\pi}}\int_0^{\infty} J_0(x)dx = \sqrt{\frac{2}{\pi}}.$$

Substituting A into $F(\omega)$ gives

$$\mathcal{F}\{J_0(x)\} = F(\omega) = \sqrt{\frac{2}{\pi}}\frac{1}{(1-\omega^2)^{1/2}}H(1-|\omega|),$$

where the Heaviside unit step function $H(1-|\omega|)$ is necessary because of the restriction imposed by the real logarithmic function that requires ω to be such that $0 < \omega^2 < 1$. ■

When working with Fourier integrals, as with the Laplace transform, it is useful to introduce the convolution operation to establish the relationship between the functions $f(x)$ and $g(x)$ and their respective Fourier transforms $F(\omega)$ and $G(\omega)$.

The **convolution** of functions $f(x)$ and $g(x)$ denoted by $f * g$ is a function of x, and if the dependence on a variable x in the convolution is to be emphasized,

it is then denoted by $(f * g)(x)$. The convolution of $f(x)$ and $g(x)$ is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt = \int_{-\infty}^{\infty} f(x-t)g(t)dt. \tag{29}$$

A slightly different definition of the convolution operation for the Fourier transform is also to be found in the literature, where it is defined as

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)g(x-t)dt.$$

When this definition is employed, the form taken by the next theorem (the convolution theorem for Fourier transforms) will require modification. This is because its form will depend on the factor $1/\sqrt{2\pi}$ and the way the constant 2π enters in the definition of the Fourier transform that is used.

THEOREM 10.6

relating the convolution of $f(x)$ and $g(x)$ and the product of their transforms

The convolution theorem for Fourier transforms Let the functions $f(x)$ and $g(x)$ be piecewise continuous, bounded, and absolutely integrable over $(-\infty, \infty)$ with the respective Fourier transforms $F(\omega)$ and $G(\omega)$. Then

(a) $$\mathcal{F}\{(f * g)(x)\} = 2\pi\, \mathcal{F}\{f(x)\}\mathcal{F}\{g(x)\}, \text{ or } \mathcal{F}\{f * g\} = 2\pi\, F(\omega)G(\omega)$$

and, conversely,

(b) $$(f * g)(x) = \sqrt{2\pi} \int_{-\infty}^{\infty} F(\omega)G(\omega)e^{i\omega x}d\omega.$$

Proof **(a)** By definition,

$$\mathcal{F}\{(f * g)(x)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)g(x-t)e^{-i\omega x}dt\right]dx$$
$$= \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t)g(x-t)e^{-i\omega x}dx\right]dt,\right.$$

where the second result follows from the first by a change in the order of integration. If we set $v = x - t$, this becomes

$$\mathcal{F}\{(f * g)(x)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[f(t)g(v)e^{-i\omega(t+v)}dt\right]dv$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \int_{-\infty}^{\infty} g(v)e^{-i\omega v}dv.$$

However, t and v are dummy variables and so may be replaced by x, causing the preceding result to become

$$\mathcal{F}\{(f * g)(x)\} = \mathcal{F}\{f(x)\}2\pi\mathcal{F}\{g(x)\},$$

showing that

$$\mathcal{F}\{(f * g)(x)\} = 2\pi\, \mathcal{F}\{f(x)\}\, \mathcal{F}\{g(x)\}, \quad \text{or} \quad \mathcal{F}\{(f * g)(x)\} = 2\pi\, F(\omega)G(\omega).$$

Result (b) follows directly from the last result by taking the inverse Fourier transform that causes a factor $\sqrt{2\pi}$ to cancel. ■

EXAMPLE 10.8

It was shown in Example 10.3(a) that the function $f(x)=\begin{cases}1, & |x|<a\\ 0, & |x|>a\end{cases}$ has the Fourier transform $F(\omega) = \sqrt{\frac{2}{\pi}}\left(\frac{\sin \omega a}{\omega}\right)$, so by the convolution theorem it follows that

$$\mathcal{F}\{(f * f)(x)\} = \sqrt{2\pi}\left[\sqrt{\frac{2}{\pi}}\left(\frac{\sin \omega a}{\omega}\right)\right]^2 = 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin^2 \omega a}{\omega^2}\right).$$

Confirm this result by calculating $(f * f)(x)$ and finding its Fourier transform.

Solution In terms of the Heaviside unit step function we can write $f(t) = H(a - |t|)$ and $f(x - t) = H(a - |x - t|)$, after which consideration of the product $f(t)f(x - t)$ shows that

$$f(t)f(x-t) = \begin{cases}1, & -a < t < x + a, (-2a < x < 0)\\ 0, & \text{otherwise}\end{cases}$$

and

$$f(t)f(x-t) = \begin{cases}1, & x - a < t < a, (0 < x < 2a)\\ 0, & \text{otherwise.}\end{cases}$$

The required convolution is then given by

$$(f * f)(x) = \begin{cases}\int_{-a}^{x+a} dt = 2a + x, (-2a < x < 0)\\ \int_{x-a}^{a} dt = 2a - x, (0 < x < 2a)\end{cases} \quad \text{and} \quad (f * f)(x) = 0 \text{ otherwise.}$$

Taking the Fourier transform of $(f^* f)(x)$, we have

$$\begin{aligned}\mathcal{F}\{(f * f)(x)\} &= \frac{1}{\sqrt{2\pi}}\left\{\int_{-2a}^{0}(2a + x)e^{-i\omega x}dx + \int_{0}^{2a}(2a - x)e^{-i\omega x}dx\right\}\\ &= \sqrt{\frac{2}{\pi}}\left(\frac{1 - \cos 2\omega a}{\omega^2}\right),\end{aligned}$$

but $1 - \cos 2\omega a = 2\sin^2 \omega a$, so

$$\mathcal{F}\{(f * f)(x)\} = 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin^2 \omega a}{\omega^2}\right),$$

as required. ■

THEOREM 10.7

the Parseval relation extended to Fourier transforms

The Parseval relation for the Fourier transform If $f(x)$ has the Fourier transform $F(\omega)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega.$$

Proof Setting $x = 0$ in result (b) of the convolution theorem gives

$$\int_{-\infty}^{\infty} f(t)g(-t)dt = \int_{-\infty}^{\infty} F(\omega)G(\omega)d\omega.$$

As the Fourier transform is defined for both real and complex functions, it follows from the definition of the transform that $\mathcal{F}\{\bar{f}(-x)\} = \bar{F}(\omega)$, where the bar indicates

complex conjugation. If we set $g(t) = \bar{f}(-t)$, the preceding result becomes

$$\int_{-\infty}^{\infty} f(t)\bar{f}(t)dt = \int_{-\infty}^{\infty} F(\omega)\bar{F}(\omega)d(\omega),$$

or

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega,$$

and the result is proved. ■

EXAMPLE 10.9

Using the result of Example 10.3(a) and the Parseval relation, show that

$$\int_{-\infty}^{\infty} \frac{\sin^2 \omega a}{\omega^2} d\omega = \pi a.$$

Solution Substituting $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ and the corresponding Fourier transform $F(\omega) = \sqrt{\frac{2}{\pi}}\left(\frac{\sin \omega a}{\omega}\right)$ found in Example 10.3(a) into the Parseval relation gives

$$\int_{-a}^{a} 1^2 dx = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2 \omega a}{\omega^2}\right)d\omega, \text{ and so } 2a = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2 \omega a}{\omega^2}\right)d\omega \ (a > 0),$$

from which the required result follows. ■

The final theorem describes the effect on the Fourier transform of $f(x)$ caused by scaling x by a factor a, shifting x by a and shifting ω by λ.

THEOREM 10.8

some useful properties of Fourier transforms

Fourier transforms involving scaling x by a, shifting x by a, and shifting ω by λ If $f(x)$ has a Fourier transform $F(\omega)$, then

(i) $\mathcal{F}\{f(ax)\} = \frac{1}{a}F(\omega/a) \quad (a > 0)$

(ii) $\mathcal{F}\{f(x-a)\} = e^{-i\omega a}F(\omega)$

(iii) $\mathcal{F}\{e^{i\lambda x}f(x)\} = F(\omega - \lambda)$

Proof As the results of the theorem follow immediately from the definition of the Fourier transform, only result (i) will be proved, and the derivation of results (ii) and (iii) left as exercises. Starting from the definition of $\mathcal{F}\{f(ax)\}$ and making the variable change $u = ax$ we have

$$\mathcal{F}\{f(ax)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{-i\omega x}dx = \frac{1}{a\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-i\omega u/a}du$$

$$= \frac{1}{a}F(\omega/a)(a > 0).$$ ■

EXAMPLE 10.10

Using the function $f(x)$ and its Fourier transform $F(\omega)$ from Example 10.9, find (a) $\mathcal{F}\{f(2x)\}$, (b) $\mathcal{F}\{f(x-\pi)\}$, and (c) $\mathcal{F}\{e^{ix}f(x)\}$.

Solution Using the results of Theorem 10.8 we have:

(a) $\mathcal{F}\{f(2x)\} = \frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\frac{\sin(\omega a/2)}{(\omega/2)}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{\sin(\omega a/2)}{\omega}\right)$

(b) $\mathcal{F}\{f(x-\pi)\} = e^{-i\pi\omega}\sqrt{\frac{2}{\pi}}\left(\frac{\sin \omega a}{w}\right)$

(c) $\mathcal{F}\{e^{ix} f(x)\} = \sqrt{\frac{2}{\pi}}\left(\frac{\sin(\omega-1)a}{\omega-1}\right)$ ■

the Dirac delta function and the Fourier transform

The **Dirac delta function** $\delta(x)$ was introduced in connection with the Laplace transform, where it was recognized that it is not a function in the usual sense, but an *operation* that only has meaning when it appears in the integrand of a definite integral. Because of its many uses in connection with physical problems described by differential equations, we now extend its definition in a way that is suitable for use with Fourier transforms. This is accomplished by defining $\delta(x-a)$ in a symmetrical manner about $x = a$ in terms of the integrals

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)dx = \int_{-\infty}^{\infty} \delta(a-x) f(x)dx = f(a), \tag{30}$$

where a is any real number.

This definition allows the Fourier transform of $\delta(x-a)$ to be represented as

$$\mathcal{F}\{\delta(x-a)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x-a)e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}}e^{-i\omega a}. \tag{31}$$

EXAMPLE 10.11 Find the Fourier transform of $f(x) = \delta(x-a)\exp[-b^2x^2]$ $(b > 0)$.

Solution By definition

$$\begin{aligned}\mathcal{F}\{\delta(x-a)\exp[-b^2x^2]\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x-a)\exp[-b^2x^2]e^{-i\omega x}dx \\ &= \frac{1}{\sqrt{2\pi}}\exp[-(a^2b^2 + i\omega a)].\end{aligned}$$ ■

Fourier Transforms of Partial Derivatives with Respect to *x* of a Function *f*(*x*, *t*) of Two Independent Variables

transforming partial derivatives

The Fourier transform with respect to x of a function $f(x,t)$ of two independent variables x and t, denoted by $F(\omega, t)$, is defined as

$${}_x\mathcal{F}\{f(x,t)\} = F(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x,t)e^{-i\omega x}dx, \tag{32}$$

where the prefix suffix x shows the variable that is being transformed.

In (32) the variable t is not involved in the integration with respect to x, so it follows that the integral by which $f(x,t)$ is recovered from $F(\omega,t)$ and the transform of partial derivatives of $f(x,t)$ with respect to x obey the same rules as those for the function of a single variable $f(x)$. Thus, the inversion integral is given by

$$f(x,t) = {}_x\mathcal{F}^{-1}\{F(\omega,t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega,t)e^{i\omega x}\,d\omega, \tag{33}$$

and the Fourier transforms of the partial derivatives of $f(x,t)$ with respect to x are given by

$${}_x\mathcal{F}\left\{\frac{\partial^n}{\partial x^n}[f(x,t)]\right\} = (i\omega)^n F(\omega,t) \tag{34}$$

$${}_x\mathcal{F}\{x^n f(x,t)\} = i^n \frac{\partial^n}{\partial\omega^n}[F(\omega,t)] \tag{35}$$

$${}_x\mathcal{F}\left\{x^m \frac{\partial^n}{\partial x^n}[f(x,t)]\right\} = i^{m+n}\frac{\partial^m}{\partial\omega^m}[\omega^n F(\omega,t)]. \tag{36}$$

These results are necessary when using the Fourier transform to solve partial differential equations involving a function $f(x,t)$ of two independent variables x and t where $-\infty < x < \infty$. Once the partial differential equation has been transformed, it becomes an ordinary differential equation for $F(\omega,t)$, with t as the independent variable and ω as a parameter. When $F(\omega,t)$ has been found by solving the differential equation, the solution $f(x,t)$ of the partial differential equation is recovered from $F(\omega,t)$ by means of the inversion integral (33).

an application to the heat equation

To illustrate the application of the Fourier transform to a partial differential equation we take as an example the **one-dimensional heat equation**, the derivation of which can be found in Section 18.5. This same partial differential equation was used when developing applications of the Laplace transform in Chapter 7. The heat equation that determines the one-dimensional temperature distribution $T(x,t)$ on a plane $x =$ constant at time t in an infinite block of metal with heat conduction properties characterized by the constant κ is given by

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t}.$$

The problem we now consider is finding the temperature distribution throughout the metal at a time t when at $t = 0$ the one-dimensional temperature distribution throughout the block is given by

$$T(x,0) = f(x),$$

where $f(x)$ is a prescribed function. Our objective will be to find the temperature $T(x,t)$ on a plane $x =$ constant at a time $t > 0$ caused by the redistribution of heat as time increases.

The Laplace transform cannot be used because when applied to the spatial variable x it is only valid for $x \geq 0$, so instead we must make use of the Fourier transform with respect to x because this applies for $-\infty \leq x \leq \infty$. Taking the Fourier transform of the heat equation with respect to x gives

$${}_x\mathcal{F}\left\{\frac{\partial^2 T}{\partial x^2}\right\} = {}_x\mathcal{F}\frac{1}{\kappa}\left\{\frac{\partial T}{\partial t}\right\},$$

so if we apply (34) with $n = 2$, while regarding ω as a parameter, this becomes

$$-\omega^2\kappa F(\omega,t) = \frac{d}{dt}[F(\omega,t)], \quad \text{where} \quad F(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} T(x,t)e^{-i\omega x}dx.$$

The transform $F(\omega,t)$ satisfies the ordinary differential equation

$$F' + \omega^2\kappa F = 0,$$

with the solution

$$F(\omega,t) = A(\omega)\exp\{-\omega^2\kappa t\},$$

where $A(\omega)$ is to be determined (remember that ω is a constant with respect to t).

As

$$F(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} T(x,t)e^{-i\omega x}dx,$$

it follows from the initial condition that

$$F(\omega,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx,$$

but $F(\omega,0) = A(\omega)$, so

$$F(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')\exp\{-i\omega x' - \omega^2\kappa t\}dx',$$

where to avoid confusion in the next step of the calculation the dummy variable x has been replaced by x'.

Applying the inversion integral to this result gives

$$\begin{aligned} T(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\{i\omega x\}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')\exp\{-i\omega x' - \omega^2\kappa t\}dx'\right]d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')\left[\int_{-\infty}^{\infty}\exp\{i\omega(x-x') - \omega^2\kappa t\}d\omega\right]dx'. \end{aligned}$$

We show separately that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{i\omega(x-x') - \omega^2\kappa t\}d\omega = \sqrt{\frac{1}{4\pi\kappa t}}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\},$$

so the required solution is seen to be given by

$$T(x,t) = \sqrt{\frac{1}{4\pi\kappa t}}\int_{-\infty}^{\infty} f(x')\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\}dx'.$$

OPTIONAL To show that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{i\omega(x-x') - \omega^2\kappa t\}d\omega = \sqrt{\frac{1}{4\pi\kappa t}}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\}$$

we need to use a complex analysis method from Chapter 15. However, before we can use this technique, the integrand of the integral on the left must be rewritten. We multiply the exponential function by e^Pe^{-P} (that is, by 1), where P is to be determined later, and as a result obtain

$$\exp\{i\omega(x-x') - \omega^2\kappa t\} = e^P\exp\{-P + i\omega(x-x') - \omega^2\kappa t\}.$$

We now choose P so that the exponent in the exponential can be expressed in the form $-(\alpha - i\beta\omega)^2$. When this is done it turns out that

$$\alpha = -\frac{i(x-x')}{2\sqrt{\kappa t}}, \quad \beta = i\sqrt{\kappa t}, \quad \text{and} \quad P = -\frac{(x-x')^2}{4\kappa t},$$

so

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{i\omega(x-x') - \omega^2\kappa t\}d\omega$$
$$= \frac{1}{2\pi}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\}\int_{-\infty}^{\infty}\exp\left\{-\left(-\frac{i(x-x')}{2\sqrt{\kappa t}} + \omega\sqrt{\kappa t}\right)^2\right\}d\omega$$

Making the change of variable

$$u = -\frac{i(x-x')}{2\sqrt{\kappa t}} + \omega\sqrt{\kappa t},$$

we find that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{i\omega(x-x') - \omega^2\kappa t\}d\omega$$
$$= \frac{1}{2\pi}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\}\frac{1}{\sqrt{\kappa t}}\int_{ic-\infty}^{ic+\infty}\exp\{-u^2\}du,$$

where $c = (x-x')^2/\sqrt{(4\kappa t)}$. If we integrate $\exp\{-u^2\}$ around the rectangle with corners located at $-R$, R, $R+ic$, and $-R+ic$ in the complex plane, and proceed to the limit as $R \longrightarrow \infty$, it follows that the integrals from $-R$ to $-R+ic$ and from R to $R+ic$ vanish, so as $\exp\{-u^2\}$ has no poles inside the rectangle, we have

$$\int_{ic-\infty}^{ic+\infty}\exp\{-u^2\}du = \int_{-\infty}^{\infty}\exp\{-u^2\}du.$$

The integral on the right is related to the error function $\mathrm{erf}(v)$ because

$$\int_0^v \exp\{-u^2\}du = \frac{\sqrt{\pi}}{2}\mathrm{erf}(v),$$

where $\mathrm{erf}(-v) = -\mathrm{erf}(v)$ and $\mathrm{erf}\,(\infty) = 1$.
Thus,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{i\omega(x-x') - \omega^2\kappa t\}d\omega$$
$$= \frac{1}{2\pi}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\}\frac{1}{\sqrt{\kappa t}}\frac{\sqrt{\pi}}{2}[\mathrm{erf}(\infty) - \mathrm{erf}(-\infty)]$$
$$= \frac{1}{2\pi}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\}\frac{1}{\sqrt{\kappa t}}\frac{\sqrt{\pi}}{2}2$$
$$= \sqrt{\frac{1}{4\pi\kappa t}}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\},$$

so we have shown that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{i\omega(x-x') - \omega^2\kappa t\}d\omega = \sqrt{\frac{1}{4\pi\kappa t}}\exp\left\{-\frac{(x-x')^2}{4\kappa t}\right\}. \tag{37}$$

Fourier integrals are discussed in references [4.3] and [4.4]. Tables of Fourier transform pairs are given in references [4.2] and [3.11].

Summary

The Fourier transform was introduced and its most important operational properties were established. The transforms of derivatives and partial derivatives were considered, and applications were made to functions defined by an ordinary differential equation and also to the unsteady one-dimensional heat equation. Partial differential equations such as the heat equation, and the use of integral transforms in their solution, will be considered in more detail in Chapter 18.

TABLE 10.1 Fourier Transform Pairs

$f(x)$	$F(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$
1. $af(x) + bg(x)$	$aF(\omega) + bG(\omega)$
2. $f^{(n)}(x)$	$(i\omega)^n F(\omega)$
3. $x^n f(x)$	$(i)^n \dfrac{d^n}{d\omega^n}[F(\omega)]$
4. $x^m f^{(n)}(x)$	$(i)^{m+n} \dfrac{d^m}{d\omega^m}[\omega^n F(\omega)]$
5. $f(ax) (a > 0)$	$\dfrac{1}{a} F(\omega/a)$
6. $f(x - a)$	$e^{-i\omega a} F(\omega)$
7. $e^{i\lambda x} f(x)$	$F(\omega - \lambda)$
8. $(f * g)(x)$	$\sqrt{2\pi}\, F(\omega) G(\omega)$ (convolution theorem)
9. $\displaystyle\int_{-\infty}^{\infty} \lvert f(x)\rvert^2 dx$	$\displaystyle\int_{-\infty}^{\infty} \lvert F(\omega)\rvert^2 d\omega$ (Parseval relation)
10. $\begin{cases} 1, & \lvert x\rvert < a \\ 0, & \lvert x\rvert > a \end{cases} \quad (a > 0)$	$\sqrt{\dfrac{2}{\pi}} \left(\dfrac{\sin a\omega}{\omega}\right)$
11. $\dfrac{\sin ax}{x} \quad (a > 0)$	$\begin{cases} \sqrt{\dfrac{\pi}{2}}, & \lvert\omega\rvert < a \\ 0, \lvert\omega\rvert > a \end{cases}$
12. $\begin{cases} 1, & a < x < b \\ 0, & \text{otherwise} \end{cases} \quad (0 < a < b)$	$\dfrac{1}{\sqrt{2\pi}} \left(\dfrac{e^{-ia\omega} - e^{-ib\omega}}{i\omega}\right)$
13. $\begin{cases} a - \lvert x\rvert, & \lvert x\rvert < a \\ 0, & \lvert x\rvert > a \end{cases}$	$\sqrt{\dfrac{2}{\pi}} \left(\dfrac{1 - \cos\omega a}{\omega^2}\right)$
14. $\dfrac{1}{a^2 + x^2} \; (a > 0)$	$\sqrt{\dfrac{\pi}{2}} \dfrac{e^{-a\lvert\omega\rvert}}{a}$
15. $\begin{cases} e^{-ax}, & x > 0 \\ 0, & x < 0 \end{cases} \; (a > 0)$	$\dfrac{1}{\sqrt{2\pi}} \left(\dfrac{1}{a + i\omega}\right)$
16. $\begin{cases} e^{ax}, & b < x < c \\ 0, & \text{otherwise} \end{cases} \; (a > 0)$	$\dfrac{1}{\sqrt{2\pi}} \left[\dfrac{e^{(a-i\omega)c} - e^{(a-i\omega)b}}{a - i\omega}\right]$

(*continued*)

TABLE 10.1 (*continued*)

$f(x)$	$F(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$
17. $e^{-a\lvert x\rvert}$ $(a > 0)$	$\sqrt{\dfrac{2}{\pi}}\left(\dfrac{a}{a^2+\omega^2}\right)$
18. $xe^{-a\lvert x\rvert}$ $(a > 0)$	$-\sqrt{\dfrac{2}{\pi}}\dfrac{2ia\omega}{(a^2+\omega^2)^2}$
19. $\begin{cases} e^{iax}, & \lvert x\rvert < b \\ 0, & \lvert x\rvert > b \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\left(\dfrac{\sin b(\omega - a)}{\omega - a}\right)$
20. $\exp(-a^2x^2)$ $(a > 0)$	$\dfrac{1}{a\sqrt{2}}\exp\left\{-\dfrac{\omega^2}{4a^2}\right\}$
21. $\begin{cases} e^{-x}x^a, & x > 0 \\ 0, & x \le 0 \end{cases}$	$\dfrac{\Gamma(a)}{\sqrt{2\pi}(1+i\omega)^a}$
22. $J_0(ax)$ $(a > 0)$	$\sqrt{\dfrac{2}{\pi}}\dfrac{H(a - \lvert\omega\rvert)}{(a^2-\omega^2)^{1/2}}$
23. $\delta(x - a)$ $(a$ real$)$	$\dfrac{1}{\sqrt{2\pi}}e^{-ia\omega}$

EXERCISES 10.2

In Exercises 1 through 10 establish the Fourier transform of the stated entry in Table 10.1.

1. Entry 11.
2. Entry 12.
3. Entry 13.
4. Entry 15.
5. Entry 16.
6. Entry 17.
7. Entry 18.
8. Entry 19.
9. Entry 21.
10. Entry 22, by using the fact that $f(x) = J_0(ax)$ satisfies the Bessel's differential equation of order zero

$$xf'' + f' + a^2xf = 0 \quad (a > 0),$$

together with the standard result $\int_0^\infty J_0(ax)dx = 1/a$.

11. Use integration by parts to show that if $f(x)$ has a finite jump discontinuity at $x = a$, then $\mathcal{F}\{f'(x)\} = i\omega F(\omega) - \frac{1}{\sqrt{2\pi}}[f(a+) - f(a-)]e^{-iwa}$.

12. (a) Use the result of Exercise 11 to find the Fourier transform of $f'(x)$ given that

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(b) Calculate $f'(x)$ and use entry 12 of Table 10.1 to find $\mathcal{F}\{f'(x)\}$ directly. Hence, show that the result obtained by this direct method is in agreement with the Fourier transform found in (a). So $f'(x) = -\delta(x-1) + \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$.

10.3 Fourier Cosine and Sine Transforms

The Fourier *cosine and sine transforms* arise as special cases of the Fourier transform, according to whether $f(x)$ is even or odd. Let us start by considering the Fourier cosine transform of $f(x)$ that can be defined when $f(x)$ is an even function that is absolutely integrable over $(-\infty, \infty)$, and so possesses a Fourier transform. Result (22) of Section 10.2 can be written

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\{\cos \omega x - i \sin \omega x\}dx, \tag{38}$$

but if $f(x)$ is an even function, the product $f(x)\cos\omega x$ is also even, so its integral over $(-\infty, \infty)$ does not vanish, though the product $f(x)\sin\,\omega x$ is an odd function, so its integral over $(-\infty, \infty)$ vanishes, causing (38) to simplify to

$$F_C(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos\omega x dx.$$

If we use the result $f(-x) = f(x)$ to change the interval of integration to $[0, \infty)$ this last result becomes

$$F_C(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos\omega x dx, \tag{39}$$

Fourier sine and cosine transforms

where the integral on the right is called the **Fourier cosine transform** of $f(x)$, and to distinguish it from the ordinary Fourier transform we write $\mathcal{F}_C\{f(x)\} = F_C(\omega)$. The **Fourier cosine inversion integral** corresponding to equation (23) of Section 10.2 becomes $f(x) = \mathcal{F}_C^{-1}\{F_C(\omega)\}$, where

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_C(\omega)\cos\omega x d\omega. \tag{40}$$

inversion integrals

A similar argument applied to (16) of Section 10.2 when $f(x)$ is an odd function leads to the result

$$F_S(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin\omega x dx, \tag{41}$$

where the integral on the right is called the **Fourier sine transform** of $f(x)$ and we write $\mathcal{F}_S\{f(x)\} = F_S(\omega)$. The corresponding **Fourier cosine inversion integral** becomes $f(x) = \mathcal{F}_S^{-1}\{F_S(\omega)\}$, where

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_S(\omega)\sin\omega x d\omega. \tag{42}$$

The Fourier cosine transform of $f(x)$ in (39) only involves $f(x)$ for $x \geq 0$, though it was derived from the Fourier transform on the assumption that $f(x)$ was an even function defined for all x. Consequently, taking the Fourier cosine transform of an arbitrary function $f(x)$ defined for $x \geq 0$ is equivalent to transforming an *even* function $f_e(x)$ obtained from $f(x)$ by setting $f_e(x) = f(x)$ for $x \geq 0$ and defining $f_e(x)$ for $x < 0$ by $f_e(-x) = f(x)$. Similarly, the Fourier sine transform of $f(x)$ in (41) only involves $f(x)$ for $x \geq 0$, though it was derived on the assumption that $f(x)$ was an odd function. So, taking the Fourier sine transform of an arbitrary function $f(x)$ defined for $x \geq 0$ is equivalent to transforming *odd* function $f_o(x)$ obtained from $f(x)$ by setting $f_o(x) = f(x)$ for $x \geq 0$ and defining $f_e(x)$for $x < 0$ by $f_e(-x) = -f(x)$.

Because (40) and (41) have been derived from (22) of Section 10.2, it follows that whenever $f(x)$ is discontinuous, the expression on the left must be replaced by $(1/2)[f(x+0) + f(x-0)]$, because the Fourier cosine and sine transforms have the same convergence properties as the Fourier transform.

EXAMPLE 10.12 Find $\mathcal{F}_C\{e^{-ax}\}$ and $\mathcal{F}_S\{e^{-ax}\}$ when $a > 0$, and use the results with the Fourier cosine and sine inversion integrals and an interchange of variables to show that

$$\mathcal{F}_C\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}\frac{e^{-a\omega}}{a} \quad \text{and} \quad \mathcal{F}_S\left\{\frac{x}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}e^{-a\omega}.$$

Solution By definition

$$\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos\omega x dx$$

$$= \text{Re}\sqrt{\frac{2}{\pi}}\left\{\int_0^\infty e^{-ax}e^{i\omega x}dx\right\} = \sqrt{\frac{2}{\pi}}\,\text{Re}\left\{\frac{1}{a-i\omega}\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{\omega^2+a^2}\right).$$

Similarly,

$$\mathcal{F}_S\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin\omega x dx$$

$$= \text{Im}\sqrt{\frac{2}{\pi}}\left\{\int_0^\infty e^{-ax}e^{i\omega x}dx\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{\omega}{\omega^2+a^2}\right).$$

Using these results in the Fourier cosine and sine inversion integrals gives

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos\omega x}{\omega^2+a^2}d\omega = \frac{2}{\pi}\int_0^\infty \frac{\omega\sin\omega x}{\omega^2+a^2}d\omega, \text{ for } a > 0,$$

so after x and ω are interchanged, these results become

$$e^{-a\omega} = \frac{2a}{\pi}\int_0^\infty \frac{\cos\omega x}{x^2+a^2}dx = \frac{2}{\pi}\int_0^\infty \frac{x\cos\omega x}{x^2+a^2}dx.$$

However,

$$\mathcal{F}_C\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos\omega x}{x^2+a^2}dx \quad \text{and} \quad \mathcal{F}_S\left\{\frac{x}{x^2+a^2}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x\sin\omega x}{x^2+a^2}dx,$$

so combining results gives

$$\mathcal{F}_C\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}\frac{e^{-a\omega}}{a} \quad \text{and} \quad \mathcal{F}_S\left\{\frac{x}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}e^{-a\omega}.$$ ■

THEOREM 10.9 **Linearity of the Fourier cosine and sine transforms** Let the functions $f(x)$ and $g(x)$ have Fourier cosine and sine transforms, and let a and b be arbitrary constants. Then

$$\mathcal{F}_C\{af(x)+bg(x)\} = a\,\mathcal{F}_C\{f(x)\} + b\,\mathcal{F}_C\{g(x)\} = aF_C(\omega) + bG_C(\omega)$$

and

$$\mathcal{F}_S\{af(x)+bg(x)\} = a\,\mathcal{F}_S\{f(x)\} + b\,\mathcal{F}_S\{g(x)\} = aF_S(\omega) + bG_S(\omega).$$

Proof The linearity properties of the Fourier cosine and sine transforms follow directly from the linearity property of the Fourier transform from which they are derived. ■

linearity of sine and cosine transforms and the transformation of derivatives

The expressions for the Fourier cosine and sine transforms of derivatives of a function $f(x)$ are slightly more complicated than those for the Fourier transform because they involve the initial values of the function and its derivatives.

THEOREM 10.10

Fourier cosine and sine transforms of derivatives Let $f(x)$ be continuous and absolutely integrable over $[0, \infty)$ and such that $\lim_{x\to\infty} f(x) = 0$. Then if $f'(x)$ and $f''(x)$ are piecewise continuous on each finite subinterval of $[0, \infty)$,

(i) $$\mathcal{F}_C\{f'(x)\} = \omega\mathcal{F}_S\{f(x)\} - \sqrt{\frac{2}{\pi}}f(0)$$

(ii) $$\mathcal{F}_S\{f'(x)\} = -\omega\mathcal{F}_C\{f(x)\}$$

(iii) $$\mathcal{F}_C\{f''(x)\} = -\omega^2\mathcal{F}_C\{f(x)\} - \sqrt{\frac{2}{\pi}}f'(0)$$

(iv) $$\mathcal{F}_S\{f''(x)\} = -\omega^2\mathcal{F}_S\{f(x)\} + \sqrt{\frac{2}{\pi}}\omega f(0).$$

Proof The proof of each result is similar, so only result (i) will be derived in detail and outlines given for the proofs of the remaining results. To obtain (i) we integrate by parts and make use of the definition of $\mathcal{F}_C\{f(x)\}$ as follows:

$$\begin{aligned}\mathcal{F}_C\{f'(x)\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos\omega x dx\\ &= \sqrt{\frac{2}{\pi}}\left[f(x)\cos\omega x\Big|_0^\infty + \omega\int_0^\infty f(x)\sin\omega x dx\right]\\ &= -\sqrt{\frac{2}{\pi}}f(0) + \omega\mathcal{F}_S\{f(x)\}.\end{aligned}$$

Result (iii) follows from (i) by replacing f by f'. Result (ii) follows in similar fashion, and (iv) follows from (ii) by replacing f by f'. ■

When Theorem 10.10 is used in the solution of second order differential equations, the initial conditions involved will help decide whether to use the cosine or sine transform. Thus, for example, if $f(0)$ is given but $f'(0)$ is unknown, the Fourier sine transform should be used to transform $f''(x)$ because result (iv) does not involve $f'(0)$. Conversely, if $f(0)$ is unknown but $f'(0)$ is given, then the Fourier cosine transform should be used to transform $f''(x)$, because result (iii) does not involve $f(0)$.

The Fourier cosine and sine transforms have Parseval relations that are analogous to the Parseval relation for the Fourier transform given in Theorem 10.7. To arrive at the first of these results we consider two functions $f(x)$ and $g(x)$ with the respective Fourier cosine transforms $F_C(\omega)$ and $G_C(\omega)$ and, using the definition of $G_C(\omega)$, write

$$\int_0^\infty F_C(\omega)G_C(\omega)\cos\omega x d\omega = \sqrt{\frac{2}{\pi}}\int_0^\infty F_C(\omega)\cos\omega x d\omega\int_0^\infty g(x)\cos\omega x dx.$$

Changing the order of integration in the expression on the right gives

$$\begin{aligned}
\sqrt{\frac{2}{\pi}}\int_0^\infty F_C(\omega)\cos\omega x\,d\omega \int_0^\infty g(v)\cos\omega v\,dv \\
= \sqrt{\frac{2}{\pi}}\int_0^\infty g(x)dx \int_0^\infty F_C(\omega)\cos\omega x\cos\omega v\,d\omega \\
= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{2}[\cos\omega(x+v)+\cos\omega|x-v|]F_C(\omega)d\omega \\
= \frac{1}{2}\int_0^\infty g(v)[f(x+v)+f(|x-v|)]dv,
\end{aligned}$$

where use has first been made of the identity $\cos u\cos v = \frac{1}{2}[\cos(u+v)+\cos(u-v)]$ and then of the Fourier cosine inversion integral.

We have established the result

$$\int_0^\infty F_C(\omega)G_C(\omega)\cos\omega x\,d\omega = \frac{1}{2}\int_0^\infty g(v)[f(x+v)+f(|x-v|)]dv.$$

Setting $x = 0$ in this last result shows that

$$\int_0^\infty F_C(\omega)G_C(\omega)d\omega = \int_0^\infty f(v)g(v)dv. \tag{43}$$

The **Parseval relation** for the **Fourier cosine transform** follows from this result by identifying $g(v)$ with $\bar{f}(v)$, for then (43) becomes

$$\boxed{\int_0^\infty |F_C(\omega)|^2 d\omega = \int_0^\infty |f(x)|^2 dx,} \tag{44}$$

where in the last integral the dummy variable v has been replaced by x.

A similar argument involving the Fourier sine transform establishes the corresponding results

$$\boxed{\int_0^\infty F_S(\omega)G_S(\omega)d\omega = \int_0^\infty f(v)g(v)dv} \tag{45}$$

and the **Parseval relation** for the **Fourier sine transform**

$$\boxed{\int_0^\infty |F_S(\omega)|^2 d\omega = \int_0^\infty |f(x)|^2 dx.} \tag{46}$$

We have arrived at the following theorem.

THEOREM 10.11

the Parseval relation extended to Fourier sine and cosine transforms

The Parseval relation for the Fourier cosine and sine transforms Let $f(x)$ have the respective Fourier cosine and sine transforms $F_C(\omega)$ and $F_S(\omega)$. Then the Parseval relation for the Fourier cosine transform is

$$\boxed{\int_0^\infty |F_C(\omega)|^2 d\omega = \int_0^\infty |f(x)|^2 dx,}$$

and the Parseval relation for the Fourier sine transform is

$$\int_0^\infty |F_S(\omega)|^2 d\omega = \int_0^\infty |f(x)|^2 dx.$$

■

Results (44) and (46) often provide a simple way of evaluating improper integrals, as shown by the following example.

EXAMPLE 10.13

Apply result (43) to $f(x) = xe^{-ax}$ and $g(x) = xe^{-bx}$, where $a > 0$, $b > 0$, given that

$$\mathcal{F}_C\{f(x)\} = \sqrt{\frac{2}{\pi}}\frac{(a^2-\omega^2)}{(a^2+\omega^2)^2} \quad \text{and} \quad \mathcal{F}_C\{g(x)\} = \sqrt{\frac{2}{\pi}}\frac{(b^2-\omega^2)}{(b^2+\omega^2)^2}.$$

Solution Substituting into (43) gives

$$\frac{2}{\pi}\int_0^\infty \frac{(a^2-\omega^2)(b^2-\omega^2)}{(a^2+\omega^2)^2(b^2+\omega^2)^2}d\omega = \int_0^\infty x^2 e^{-(a+b)x}dx,$$

and after integrating the expression on the right and multiplying by $\pi/2$ we find that

$$\int_0^\infty \frac{(a^2-\omega^2)(b^2-\omega^2)}{(a^2+\omega^2)^2(b^2+\omega^2)^2}d\omega = \frac{\pi}{(a+b)^3}.$$

This integral can be evaluated by other techniques, but the preceding method is one of the simplest. ■

The final theorem in this section is the analogue of Theorem 10.8, and it is useful when transforming known Fourier cosine and sine transforms.

THEOREM 10.12

shifting and scaling Fourier sine and cosine transforms

Shifting ω and scaling x in Fourier cosine and sine transforms Let $f(x)$ have the respective Fourier cosine and sine transforms $F_C(\omega)$ and $F_S(\omega)$. Then

(i) $\mathcal{F}_C\{\cos(ax)f(x)\} = \frac{1}{2}\{F_C(\omega+a) + F_C(\omega-a)\}$

(ii) $\mathcal{F}_C\{\sin(ax)f(x)\} = \frac{1}{2}\{F_S(a+\omega) + F_S(a-\omega)\}$

(iii) $\mathcal{F}_S\{\cos(ax)f(x)\} = \frac{1}{2}\{F_S(\omega+a) + F_S(\omega-a)\}$

(iv) $\mathcal{F}_S\{\sin(ax)f(x)\} = \frac{1}{2}\{F_C(\omega-a) - F_C(\omega+a)\}$

(v) $\mathcal{F}_C\{f(ax)\} = \dfrac{1}{a}F_C(\omega/a) \quad (a > 0)$

(vi) $\mathcal{F}_S\{f(ax)\} = \dfrac{1}{a}F_S(\omega/a) \quad (a > 0).$

Proof (i) $\mathcal{F}_C\{\cos(ax)f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos(\omega x)\cos(ax)f(x)dx$, but

$$\cos(ax)\cos(\omega x) = \frac{1}{2}[\cos\{(a+\omega)x\} + \cos\{(a-\omega)x\}],$$

so

$$\begin{aligned}\mathcal{F}_C\{\cos(ax)f(x)\} &= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty \cos\{(a+\omega)x\}f(x)dx \\ &\quad+\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty \cos\{(a-\omega)x\}f(x)dx \\ &= \frac{1}{2}\{F_C(\omega+a)+F_C(\omega-a)\}.\end{aligned}$$

Results (ii) to (iv) follow in similar fashion, whereas results (v) and (vi) follow from the definitions of the Fourier cosine and sine transforms after making the change of variable $u = ax$. ■

EXAMPLE 10.14 Given $f(x) = e^{-ax}$ with $a > 0$, use the results of Theorem 10.12 to find (a) $\mathcal{F}_C\{\cos bx f(x)\}$ and (b) $\mathcal{F}_S\{f(bx)\}$, when $b > 0$.

Solution

(a) Using Theorem 10.12 (i) with

$$\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{\omega^2+a^2}\right),$$

gives

$$\begin{aligned}\mathcal{F}_C\{\cos bx e^{-ax}\} &= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\frac{a}{(\omega+b)^2+a^2}\right)+\frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\frac{a}{(\omega-b)^2+a^2}\right) \\ &= \sqrt{\frac{2}{\pi}}\frac{a(\omega^2+a^2+b^2)}{[(\omega+b)^2+a^2][(\omega-b)^2+a^2]}.\end{aligned}$$

(b) Using Theorem 10.12 (vi) with

$$\mathcal{F}_S\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{\omega}{\omega^2+a^2}\right)$$

gives

$$\mathcal{F}_S\{f(bx)\} = \mathcal{F}_S\{e^{-abx}\} = \frac{1}{b}\sqrt{\frac{2}{\pi}}\left(\frac{\omega/b}{(\omega/b)^2+a^2}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{\omega}{\omega^2+a^2b^2}\right).$$

This result is to be expected, as it follows directly from the original result when a is replaced by ab. ■

When Fourier cosine and sine transforms are used in the solution of partial differential equations, the function to be transformed is a function of more than one variable. So, for example, the operation of taking the Fourier cosine transform of $f(x, y)$ with respect to x, denoted by $F_C(\omega, y)$, is given by

$$_x\mathcal{F}_C\{f(x, y)\} = F_C(\omega, y) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x, y)\cos\omega x dx. \tag{47}$$

Similarly, the operation of taking the Fourier sine transform of $f(x, y)$ with respect to y, denoted by $F_S(x, \omega)$, is given by

$$ {}_y\mathcal{F}_S\{f(x, y)\} = F_S(x, \omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x, y) \sin \omega y dy. \tag{48} $$

As a variable that has not been transformed only appears as a parameter in the transform, it follows immediately that the rules for transforming partial derivatives follow directly from the rules for transforming derivatives of functions of a single independent variable. As a result, when interpreted in terms of a function $f(x, y)$, the entries in Theorem 10.10 take the following form.

transform of partial derivatives by Fourier sine and cosine transforms

Fourier cosine and sine transforms of partial derivatives of a function f(x, y)

$$ {}_x\mathcal{F}_C\{f'(x, t)\} = \omega F_S(\omega, t) - \sqrt{\frac{2}{\pi}} f(0, t) \tag{49} $$

$$ {}_x\mathcal{F}_S\{f'(x, t)\} = -\omega F_C(\omega, t) \tag{50} $$

$$ {}_x\mathcal{F}_C\{f''(x, t)\} = -\omega^2 F_S(\omega, t) - \sqrt{\frac{2}{\pi}} f'(0, t) \tag{51} $$

$$ {}_x\mathcal{F}_S\{f''(x, t)\} = -\omega^2 F_S(\omega, t) + \sqrt{\frac{2}{\pi}} \omega f(0, t) \tag{52} $$

It also follows that when transforming with respect to x partial derivatives of $f(x, y)$ with respect to y, the function f is transformed and the partial derivative of $f(x, y)$ with respect to y becomes an ordinary derivative with respect to y of the transformed function. So, for example,

$$ {}_x\mathcal{F}_C\left\{\frac{\partial^n f(x, y)}{\partial y^n}\right\} = \frac{d^n F_C(\omega, y)}{dy^n}, $$

with corresponding results for mixed derivatives.

another application to the heat equation

To provide a motivation for these results we again anticipate the discussion of partial differential equations that is to follow in Chapter 18. Our objective now will be to solve the same **initial boundary value problem** for the one-dimensional **heat equation** that was solved previously by means of the Laplace transform. The one-dimensional heat equation governing the temperature $T(x, t)$ in a semi-infinite slab of metal at a distance x from its plane face at time t is

$$ \frac{\partial^2 T}{\partial x^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t}, \tag{53} $$

and as before we will seek a solution subject to the initial condition

$$ T(x, 0) = 0 \tag{54} $$

and the boundary condition

$$ T(0, t) = T_0, \quad t \geq 0. \tag{55} $$

The initial condition (54) says that at time $t = 0$ all the metal in the slab is at temperature $T = 0$, whereas the boundary condition (55) says that for $t > 0$ the

plane face of the slab of metal is suddenly maintained at the constant temperature $T = T_0$.

As an initial temperature is known, but $\partial T/\partial x$ is unknown, consideration of results (49) to (52) suggests that we use the Fourier sine transform because it is valid for $x \geq 0$ and it only requires knowledge of $T(0,\ t) = T_0$. Accordingly, taking the Fourier sine transform of (53) with $\mathcal{F}_S\{T(x,\ t)\} = T_S(\omega,\ t)$, we have

$$\mathcal{F}_S\left\{\frac{\partial^2 T}{\partial x^2}\right\} = \frac{1}{\kappa} \quad \mathcal{F}_S\left\{\frac{\partial T}{\partial t}\right\},$$

so using (52) and regarding ω as a parameter (it is independent of t), we obtain

$$\kappa\left(-\omega^2 T_S(\omega,\ t) + \omega T_0\sqrt{\frac{2}{\pi}}\right) = \frac{d}{dt}[T_S(\omega,\ t)].$$

Thus, $T_S(\omega, t)$ satisfies the linear differential equation

$$T_S' + \omega^2\kappa T_S = \omega\kappa T_0\sqrt{\frac{2}{\pi}}$$

with the solution

$$T_S(\omega,\ t) = \frac{T_0}{\omega}\sqrt{\frac{2}{\pi}} + A(\omega)\exp\{-\omega^2\kappa t\},$$

where the arbitrary function $A(\omega)$ enters as the integration "constant" when $T_S(\omega,\ t)$ is integrated with respect to t, during which ω behaves as a constant.

Applying the inverse Fourier sine transform to this last result gives

$$T(x,\ t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \left\{\frac{T_0}{\omega}\sqrt{\frac{2}{\pi}} + A(\omega)\exp\{-\omega^2\kappa t\}\right\}\sin\omega x d\omega.$$

To determine $A(\omega)$ we now apply the initial condition $T(x,\ 0) = 0$ to the preceding result, which then becomes

$$0 = \sqrt{\frac{2}{\pi}}\int_0^\infty \left\{\frac{T_0}{\omega}\sqrt{\frac{2}{\pi}} + A(\omega)\right\}\sin\omega x d\omega.$$

This must be true for all ω, but this is only possible if $A(\omega) = -\frac{T_0}{\omega}\sqrt{\frac{2}{\pi}}$, and so

$$T(x, t) = T_0\sqrt{\frac{2}{\pi}}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty \left(\frac{1 - \exp(-\kappa t\omega^2)}{\omega}\right)\sin\omega x d\omega\right\}.$$

The bracketed term is the inverse Fourier sine transform of $\{[1 - \exp(-\kappa\omega^2)]/\omega\}$, so if we use entry 17 in Table 10.3, the solution becomes

$$T(x, t) = T_0 \operatorname{erfc}\left\{\frac{x}{2\sqrt{\kappa t}}\right\}.$$

This is the result that was obtained in Section 7.3 (e) (ii) by means of the Laplace transform. The result agrees with physical intuition because for any fixed x we have $\lim_{t\to\infty} \operatorname{erfc}\{\frac{x}{2\sqrt{\kappa t}}\} = 1$, showing that as $t \to \infty$, so $T(x, t) \to T_0$ the constant temperature of the plane face of the metal.

Summary

The Fourier sine and cosine transforms were introduced, their inversion integrals were stated, and the main operational properties of the transforms were established. The sine and cosine transforms of ordinary and partial derivatives were derived and applications were made to the unsteady one-dimensional heat equation.

TABLE 10.2 Fourier Cosine Transform Pairs

$f(x)$	$F_C(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos\omega x dx$
1. $af(x) + bg(x)$	$aF(\omega) + bG(\omega)$
2. $\cos(ax)f(x)$	$\frac{1}{2}\{F_C(\omega + a) + F_C(\omega - a)\}$
3. $\sin(ax)f(x)$	$\frac{1}{2}\{F_S(a + \omega) + F_S(a - \omega)\}$
4. $f(ax)$	$\frac{1}{a}F_C\left(\frac{\omega}{a}\right)(a > 0)$
5. $f'(x)$	$\omega F_S(\omega) - \sqrt{\frac{2}{\pi}}f(0)$
6. $f''(x)$	$-\omega^2 F_C(\omega) - \sqrt{\frac{2}{\pi}}f'(0)$
7. $\int_0^\infty \lvert f(x)\rvert^2 dx$	$\int_0^\infty \lvert F(\omega)\rvert^2 d\omega$ (Parseval relation)
8. $\int_0^\infty f(x)g(x)dx$	$\int_0^\infty F_C(\omega)G_C(\omega)d\omega$
9. $\begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\left(\frac{\sin a\omega}{\omega}\right)$
10. $\begin{cases} 1, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\left(\frac{\sin b\omega - \sin a\omega}{\omega}\right)$
11. $x^{\alpha-1}(0 < \alpha < 1)$	$\sqrt{\frac{2}{\pi}}\frac{\Gamma(\alpha)}{\omega^\alpha}\cos\frac{\alpha\pi}{2}$
12. $\begin{cases} x, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\left(\frac{\cos b\omega + b\omega\sin b\omega - \cos a\omega - a\omega\sin a\omega}{\omega^2}\right)$
13. $e^{-ax}(a > 0)$	$\sqrt{\frac{2}{\pi}}\left(\frac{a}{\omega^2 + a^2}\right)$
14. $xe^{-ax}(a > 0)$	$\sqrt{\frac{2}{\pi}}\frac{(a^2 - \omega^2)}{(a^2 + \omega^2)^2}$
15. $\exp\{-ax^2\}\ (a > 0)$	$\frac{1}{\sqrt{2a}}\exp\left\{-\frac{\omega^2}{4a}\right\}$
16. $\frac{1}{x^2 + a^2}(a > 0)$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\omega}}{a}$
17. $J_0(ax)(a > 0)$	$\sqrt{\frac{2}{\pi}}\frac{H(a - \omega)}{(a^2 - \omega^2)^{1/2}}$
18. $\frac{\sin ax}{x}(a > 0)$	$\sqrt{\frac{2}{\pi}}H(a - \omega)$

TABLE 10.3 Fourier Sine Transform Pairs

$f(x)$	$F_S(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x dx$
1. $af(x) + bg(x)$	$aF(\omega) + bG(\omega)$
2. $\cos(ax) f(x)$	$\frac{1}{2}\{F_S(\omega + a) + F_S(\omega - a)\}$
3. $\sin(ax) f(x)$	$\frac{1}{2}\{F_C(\omega - a) - F_C(\omega + a)\}$
4. $f(ax)$	$\frac{1}{a} F_S\left(\frac{\omega}{a}\right) \quad (a > 0)$
5. $f'(x)$	$-\omega F_C(\omega)$
6. $f''(x)$	$-\omega^2 F_S(\omega) + \sqrt{\frac{2}{\pi}} \omega f'(0)$
7. $\int_0^\infty \lvert f(x)\rvert^2 dx$	$\int_0^\infty \lvert F(\omega)\rvert^2 d\omega$ (Parseval relation)
8. $\int_0^\infty f(x)g(x)dx$	$\int_0^\infty F_S(\omega)G_S(\omega)d\omega$
9. $\begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos a\omega}{\omega}\right)$
10. $\begin{cases} 1, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \left(\frac{\cos a\omega - \cos b\omega}{\omega}\right)$
11. $x^{\alpha-1}$ $(0 < \alpha < 1)$	$\sqrt{\frac{2}{\pi}} \frac{\Gamma(\alpha)}{\omega^\alpha} \sin \frac{\alpha\pi}{2}$
12. e^{-ax} $(a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{\omega}{(\omega^2 + a^2)}$
13. xe^{-ax} $(a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{2a\omega}{(\omega^2 + a^2)^2}$
14. $x \exp\{-ax^2\}$ $(a > 0)$	$\frac{\omega}{(2a)^{3/2}} \exp\left\{-\frac{\omega^2}{4a}\right\}$
15. $\frac{x}{x^2 + a^2}$ $(a > 0)$	$\sqrt{\frac{\pi}{2}} e^{-a\omega}$
16. $\frac{\cos ax}{x}$ $(a > 0)$	$\sqrt{\frac{\pi}{2}} H(\omega - a)$
17. $\operatorname{erfc}\left\{\frac{x}{2a}\right\}$ $(a > 0)$	$\sqrt{\frac{2}{\pi}} \left\{\frac{1 - \exp(-a^2\omega^2)}{\omega}\right\}$

EXERCISES 10.3

In Exercises 1 through 10 establish the Fourier cosine transform of the stated entry in Table 10.2.

1. Entry 9.
2. Entry 10.
3. Entry 11.
4. Entry 12.
5. Entry 13.
6. Entry 14.
7. Entry 15.
8. Entry 16.
9. Entry 17.
10. Entry 18.

In Exercises 11 through 15 find the Fourier cosine transform of the stated function.

11. $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & \text{otherwise.} \end{cases}$

12. $f(x) = \begin{cases} \cos x, & 0 \le x \le \pi \\ 0, & \text{otherwise.} \end{cases}$

13. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$

14. $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$

15. $f(x) = \begin{cases} 1 - x^2, & 0 \le x < 1 \\ 0, & \text{otherwise.} \end{cases}$

In Exercises 16 through 23 establish the Fourier sine transform of the stated entry in Table 10.3.

16. Entry 9.

17. Entry 10.

18. Entry 11.

19. Entry 12.

20. Entry 13.

21. Entry 14.

22. Entry 15.

23. Entry 16.

In Exercises 24 through 28 find the Fourier sine transform of the stated function.

24. $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & \text{otherwise.} \end{cases}$

25. $f(x) = \begin{cases} \cos x, & 0 \le x \le \pi \\ 0, & \text{otherwise.} \end{cases}$

26. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$

27. $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$

28. $f(x) = \begin{cases} 1 - x^2, & 0 \le x < 1 \\ 0, & \text{otherwise.} \end{cases}$